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Piecewise rigidity

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Abstract

In this paper we provide a Liouville type theorem in the framework of fracture mechanics, and more precisely in the theory of *SBV* deformations for cracked bodies. We prove the following rigidity result: if $u \in SBV(\Omega, \mathbb{R}^N)$ is a deformation of Ω whose associated crack J_u has finite energy in the sense of Griffith's theory (i.e., $\mathcal{H}^{N-1}(J_u) < \infty$), and whose approximate gradient ∇u is almost everywhere a rotation, then u is a collection of an at most countable family of rigid motions. In other words, the cracked body does not store elastic energy if and only if all its connected components are deformed through rigid motions. In particular, global rigidity can fail only if the crack disconnects the body.

Keywords: Rigidity; Liouville theorem; *SBV* functions; Discontinuities; Caccioppoli partitions

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1. Introduction

A classical rigidity result in nonlinear elasticity, due to Liouville, states that if an elastic body is deformed in such a way that its deformation gradient is pointwise a rotation, then the body is indeed subject to a rigid motion. If the body is supposed to be hyperelastic with an elastic energy density $\mathcal{W}$ defined on a *natural* reference configuration Ω, a standard assumption for $\mathcal{W}$ which comes from its *frame indifference* is that $\mathcal{W}$ is minimized exactly on the set of rotations SO(3). Hence the rigidity result implies that the body does not store elastic energy if and only if it is deformed through a rigid motion.

From a mathematical viewpoint, Liouville's theorem can be stated as follows: if $\Omega \subseteq \mathbb{R}^N$ is open and connected, $u \in C^\infty(\Omega; \mathbb{R}^N)$ is such that $\nabla u(x) \in \mathrm{SO}(N)$ for every $x \in \Omega$, then $u = Rx + b$ for some $b \in \mathbb{R}$ and $R \in \mathrm{SO}(N)$. The assumption on the regularity of u has been fairly weakened, and now the same rigidity result is available for deformations in the class of Sobolev maps (see Yu. Reshetnyak [19]). In this case the deformation gradient is defined only almost everywhere in Ω, so that the assumption for rigidity is $\nabla u(x) \in \mathrm{SO}(N)$ for a.e. $x \in \Omega$.

A quantitative rigidity estimate has been provided recently by Friesecke, James and Müller [14], in order to derive nonlinear plates theories from three-dimensional elasticity. They proved that if Ω is connected and with Lipschitz boundary, there exists a constant C depending only on Ω and N such that for every $u \in W^{1,2}(\Omega, \mathbb{R}^N)$

$$\min_{R \in \mathrm{SO}(N)} \|\nabla u - R\|_{L^2(\Omega)} \leqslant C \big\|\mathrm{dist}\big(\nabla u, \mathrm{SO}(N)\big)\big\|_{L^2(\Omega)}. \tag{1.1}$$

As a consequence, if the deformation gradient is close to rotations (in L^2), then it is in fact close to a unique rotation. Estimate (1.1) is indeed true in L^p for every $1 < p < +\infty$, and this can be proved with minor modification of the arguments of [14].

The aim of this paper is to discuss the problem of rigidity in the framework of fracture mechanics, that is for bodies that can not only deform elastically, but also be cracked along surfaces where the deformation becomes discontinuous. The class of admissible deformations that we consider, in this setting, will be the space of *special functions of bounded variation* $SBV(\Omega; \mathbb{R}^N)$ (see Section 2 for a precise definition). Given $u \in SBV(\Omega; \mathbb{R}^N)$, the approximate gradient ∇u (which exists at almost every point of Ω) takes into account the elastic part of the deformation, while the jump set J_u represents a crack in the reference configuration. The set J_u is rectifiable, that is, it can be covered (up to a $\mathcal{H}^{N-1}$-negligible set) by a countable number of C^1 submanifolds of $\mathbb{R}^N$. So J_u is, in some sense, an $(N-1)$-dimensional surface.

In the context of *SBV* deformations, we cannot expect a rigidity result as for elastic deformations, because a crack can divide the body into two parts, each one subject to a different rigid deformation. We prove that this is essentially the only way rigidity can be violated, provided the crack J_u has "finite energy" (which, in the framework of Griffith's theory, means that its total

$(N-1)$-dimensional surface is finite). If the body is not suitably divided by a crack in several components, then rigidity as in the elastic case holds.

In order to formulate our result, we need some notions from geometric measure theory in order to make precise the notion of a partition Ω in connection with *SBV* deformations. We refer to Section 2 for more details. We say that a partition $(E_i)_{i\in\mathbb{N}}$ of Ω is a *Caccioppoli partition* if $\sum_{i\in\mathbb{N}} P(E_i,\Omega) < +\infty$, where $P(E_i,\Omega)$ denotes the perimeter of E_i in Ω. Given a rectifiable set $K \subset \Omega$, we say that a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω is subordinated to K if (up to a $\mathcal{H}^{N-1}$-negligible set) the reduced boundary $\partial^* E_i$ of E_i is contained in K for every $i\in\mathbb{N}$. We say that $\Omega\setminus K$ is *indecomposable* if the only Caccioppoli partition subordinated to K is the trivial one, i.e., $E_0 = \Omega$.

The main rigidity result of the paper is the following Liouville's type theorem for *SBV*-deformations.

Theorem 1.1. *Let $u \in SBV(\Omega,\mathbb{R}^N)$ such that $\mathcal{H}^{N-1}(J_u) < +\infty$ and $\nabla u(x)\in \mathrm{SO}(N)$ for a.e. $x\in\Omega$. Then u consists in at most countably many rigid deformations, i.e., there exists a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ subordinated to J_u such that for a.e. $x\in\Omega$,*

$$u(x) = \sum_{i\in\mathbb{N}} (R_i x + b_i)\mathbf{1}_{E_i}(x),$$

where $R_i\in\mathrm{SO}(N)$ and $b_i\in\mathbb{R}^N$ (as a consequence, $J_u = \bigcup_{i\in\mathbb{N}} \partial^* E_i$ up to a set of $\mathcal{H}^{N-1}$ measure zero). In particular, if $\Omega\setminus J_u$ is indecomposable, then u is a rigid deformation, i.e., $u(x) = Rx + b$ for some $b\in\mathbb{R}^N$ and $R\in\mathrm{SO}(N)$ (hence, $J_u=\emptyset$).

Let us observe that the assumption that $\mathcal{H}^{N-1}(J_u)$ is finite is essential in this result. Indeed, it has been shown by Alberti [1,6] that any N-dimensional L^1 vector field can be the gradient of a suitable *SBV* function, so that the rigidity clearly fails if one just assumes $\nabla u(x)\in\mathrm{SO}(N)$ for a.e. $x\in\Omega$.

In the context of fracture mechanics, Theorem 1.2 implies the following fact. Assume that the density of the elastic energy stored in the cracked body is represented by a function $\mathcal{W}$ vanishing exactly on $\mathrm{SO}(N)$. Then a deformation u of class *SBV* does not store elastic energy if and only if the crack J_u divides Ω in several subbodies, each one subject to a rigid motion. If J_u is not enough to create subbodies of Ω, then u is a rigid motion for the entire body (and there is no jump J_u at all). In this respect, the space *SBV* seems to be appropriate for the study of elastic properties of cracked hyperelastic bodies.

The main difficulty to prove Theorem 1.1 is that the differential constraint $\operatorname{curl}\nabla u = 0$, valid for every Sobolev function, does not hold in general for *SBV* functions, because ∇u is only a part of the distributional derivative of u. However we prove that if $u\in SBV(\Omega;\mathbb{R}^N)$ with $\nabla u\in L^\infty(\Omega; M^{N\times N})$ then $\operatorname{curl}\nabla u$ is a measure, which is absolutely continuous with respect to $\mathcal{H}^{N-1}\llcorner J_u$. This result (up to our knowledge, new and interesting on its own), combined with the quantitative rigidity estimate (1.1) is enough to obtain our rigidity result.

The set of rotations in $\mathbb{R}^N$ can be replaced by any compact set of matrices $\mathcal{K}\subseteq M^{N\times N}$ which satisfy a L^p-quantitative rigidity estimate for $1 < p < \frac{N}{N-1}$, i.e., there exists $C > 0$ depending on N and p such that, for every $u \in W^{1,p}(\Omega, \mathbb{R}^N)$,

$$\min_{K \in \mathcal{K}} \|\nabla u - K\|_{L^p(\Omega)} \leqslant C \big\|\operatorname{dist}(\nabla u, \mathcal{K})\big\|_{L^p(\Omega)}. \tag{1.2}$$

Theorem 1.1 is obtained as a particular case of the following rigidity result.

Theorem 1.2 *(The rigidity result). Let $\mathcal{K} \subseteq M^{N\times N}$ be a compact set such that the quantitative rigidity estimate* (1.2) *holds for some $p \in (1, N/(N-1))$. Let $u \in SBV(\Omega, \mathbb{R}^N)$ be such that $\mathcal{H}^{N-1}(J_u) < +\infty$ and $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$. Then there exists a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω subordinated to J_u such that for a.e. $x \in \Omega$,*

$$u(x) = \sum_{i\in\mathbb{N}} (K_i x + b_i)\mathbf{1}_{E_i}(x),$$

where $K_i \in \mathcal{K}$ and $b_i \in \mathbb{R}^N$ (*as a consequence, $J_u = \bigcup_{i\in\mathbb{N}} \partial^* E_i$ up to a set of $\mathcal{H}^{N-1}$-measure zero*). *In particular if $\Omega \setminus J_u$ is indecomposable, then $u(x) = Kx + b$ for some $K \in \mathcal{K}$, $b \in \mathbb{R}^N$* (*hence, $J_u = \emptyset$*).

In order to prove Theorem 1.2, the key point is to show that ∇u is a piecewise constant function that can jump only on J_u, i.e., $\nabla u \in SBV(\Omega, M^{N\times N})$ with $\nabla(\nabla u) = 0$ and $J_{\nabla u} \subseteq J_u$: this implies that ∇u is constant on a Caccioppoli partition subordinated to J_u, and hence that u is affine on the same partition.

In order to establish that ∇u is piecewise constant with jumps on J_u, we use an approximation based on a covering argument inspired by [14]. First of all we split our domain in a disjoint union of small cubes Q_h of size h. On many of these cubes, $\mathcal{H}^{N-1}(J_u \cap Q_h)$ will be small, showing that $\operatorname{curl} \nabla u$ is close to zero in Q_h. A Sobolev type estimate for L^1 vector fields with curl-measure shows then that ∇u is close in L^p to the gradient ∇w_h of a Sobolev function, which by the quantitative rigidity estimate (1.2) is close in L^p to a unique matrix $K(Q_h) \in \mathcal{K}$. We show that ∇u is approximated by the piecewise constant functions ψ_h such that $\psi_h = K(Q_h)$ on each Q_h. The sequence $(\psi_h)_{h\in\mathbb{N}}$ has a uniformly bounded total variation which is controlled by $\operatorname{curl} \nabla u$ and so by $\mathcal{H}^{N-1} \llcorner J_u$: we prove this, as in [14], by using again the quantitative rigidity estimate on the union of neighboring cubes. An application of the compactness theorem for BV functions is then enough to get the conclusion.

Let us mention that a local version of Liouville theorem on sets of finite perimeter, for Lipschitz maps, was already given in [13]. There, Dolzmann and Müller prove that if $u : \Omega \to \mathbb{R}^N$ is in $W^{1,\infty}(\Omega; \mathbb{R}^N)$, $\det \nabla u \geqslant c > 0$, and $\nabla u \in \mathrm{SO}(N)$ for a.e. $x \in E$, where E is a subset of Ω with finite perimeter, then $\nabla u \mathbf{1}_E \in BV(\Omega)$, and $D(\nabla u \mathbf{1}_E) \llcorner (\Omega \setminus \partial^* E) = 0$. (So that the thesis of Theorem 1.1 holds inside E.) This is easily deduced from Theorem 1.1.

Rigidity results in the spirit of Liouville's theorem play also an important role in order to understand possible microstructures arising in elastic bodies. The problem of microstructures can be stated mathematically in the following way: given a set of matrices $\mathcal{K} \subseteq M^{N\times N}$, find Lipschitz mappings $u : \Omega \to \mathbb{R}^N$ such that $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$. $\mathcal{K}$ is said to be rigid if it does not admit nontrivial microstructures, i.e., if the only maps $u \in W^{1,\infty}(\Omega)$ such that $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$ are affine.

An example of rigid set of matrices is provided by a famous result by Ball and James [7]: $\mathcal{K} = \{K_1, K_2\}$ is rigid if and only if $\operatorname{rank}(K_1 - K_2) \geqslant 2$. In this case, Ball and James proved that rigidity holds also in the stronger sense of *approximate solutions*: for every sequence $(u_h)_{h\in\mathbb{N}}$ of equi-Lipschitz functions such that $\mathrm{dist}(\nabla u_h, \mathcal{K}) \to 0$ in measure, then either $\mathrm{dist}(\nabla u_h, K_1) \to 0$, or $\mathrm{dist}(\nabla u_h, K_2) \to 0$ in measure.

Theorem 1.2 can be used to infer a similar result in the framework of the discontinuous deformations of class *SBV*. The quantitative rigidity estimate we need to apply our arguments has been recently provided by De Lellis and Székelyhidi [12]: they prove in particular that if $\mathcal{K} \subseteq M^{N\times N}$ is a finite set of matrices which is rigid for approximate solutions, then the quantitative rigidity estimate (1.2) holds for any $p \in (1, +\infty)$ provided that Ω is Lipschitz-regular: Theorem 1.2 hence applies to $\mathcal{K} = \{K_1, K_2\}$ as above, and, thanks to an extension by Šverák of Ball and James' result, to $\mathcal{K}$ consisting of three matrices without any rank-1 connection [20]. (The result of De Lellis and Székelyhidi actually extends our thesis to any finite union of compact sets, each satisfying a quantitative rigidity estimate, and such that any gradient Young measure supported by the union is, in fact, supported by only one of the sets.)

For completeness, let us eventually say that if $\mathcal{K}$ consists of four matrices without any rank-1 connection, rigidity can fail for approximate solutions for a suitable choice of the involved matrices (see [21,22]), while $\mathcal{K}$ is always rigid with respect to exact solutions (see [11]). The case $N = 5$ is nicely illustrated in [18] by a non-rigid five point configuration without any rank-1 connection.

Finally, we observe that a linear version of Theorem 1.1 also holds true, with a simpler proof: a *SBD* [5,8] displacement u with $e(u) = 0$ a.e. is piecewise made of (linearized) rigid motions. (A rigid motion being, in this case, an affine displacement with antisymmetric gradient.) A detailed statement is given in Appendix A. The proof of this result relies on the same ideas as the proof of our estimate on $\mathrm{curl}\,\nabla u$ in Section 3. The latter is based on a discretization, at a step $\varepsilon > 0$, of the map u, and a reinterpolation argument that produces a "simplified" approximation u_ε of u, regular enough to be able to explicitly compute its curl. We then pass to the limit as $\varepsilon \to 0$. When $e(u) = 0$, one shows that a slightly more complex (but similar) approach yields approximate functions with $e(u_\varepsilon) = 0$, as well. It follows that each u_ε is piecewise affine on a finite perimeter partition, and this property is conserved in the limit. On the other hand, the nonlinear assumption $\nabla u \in \mathrm{SO}(N)$ does not yield much more than the fact that u_ε is Lipschitz away from its jump set, and it is not clear what to deduce in the limit. It seems thus that such a direct approach fails to work in this case (hence the need for quantitative rigidity estimates). If u were locally invertible—as in the already mentioned rigidity result of Dolzmann and Müller [13]—we might be able to show some rigidity of the discretized u_ε (modulo, of course, regularity issues) and then pass to the limit: but this is of course not the case.

The paper is organized as follows. Section 2 contains mathematical preliminaries: we recall some facts from geometric measure theory and from the theory of *SBV* spaces. The proof of Theorem 1.2 is given in Section 5. It is based on estimates proven in the previous sections. A key point is to understand how far ∇u is from being a real gradient. The distribution $\mathrm{curl}\,\nabla u$ provides such an information: we show in Section 3 that when u satisfies the assumptions of the rigidity theorem, it is a Radon measure bounded by $\mathcal{H}^{N-1} \llcorner J_u$ (times a constant). A quantitative estimate of the distance to the set of "real" gradients is then obtained. It follows from a Sobolev type estimate for L^1 vector fields with curl-measure in a cube, which is shown in Section 4. In Appendix A, we discuss the linear variant of Theorem 1.1.

2. Notations and preliminaries

In this section we recall the definition of the space *SBV* and some facts from geometric measure theory that will be used throughout the paper. We refer to [6] for further details.

2.1. The space SBV

Let Ω be an open set in $\mathbb{R}^N$. We say that $u \in BV(\Omega; \mathbb{R}^N)$ if $u \in L^1(\Omega; \mathbb{R}^N)$, and its distributional derivative Du is a vector-valued Radon measure on Ω. We say that $u \in SBV(\Omega; \mathbb{R}^N)$ if $u \in BV(\Omega; \mathbb{R}^N)$ and its distributional derivative can be represented as

$$Du(A) = \int_A \nabla u(x)\, dx + \int_{A \cap J_u} \big(u^+(x) - u^-(x)\big) \otimes \nu(x)\, d\mathcal{H}^{N-1}(x),$$

where ∇u denotes the approximate gradient of u, J_u denotes the set of approximate jumps of u, u^+ and u^- are the traces of u on J_u, $\nu(x)$ is the normal to J_u at x, and $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. The symbol $\otimes$ denotes the tensorial product of vectors:

$$(a \otimes b)_{ij} = a_i b_j \quad \text{for every } a, b \in \mathbb{R}^N.$$

Note that if $u \in SBV(\Omega; \mathbb{R}^N)$, then the singular part of Du is concentrated on J_u which is a countably $\mathcal{H}^{N-1}$-rectifiable set: there exists a set E with $\mathcal{H}^{N-1}(E) = 0$ and a sequence $(M_i)_{i\in\mathbb{N}}$ of C^1-submanifolds of $\mathbb{R}^N$ such that $J_u \subseteq E \cup \bigcup_{i\in\mathbb{N}} M_i$.

We set, for $d \geqslant 1$,

$$SBV_\infty\big(\Omega; \mathbb{R}^d\big) := \big\{u \in SBV\big(\Omega; \mathbb{R}^d\big)\colon\ \nabla u \in L^\infty\big(\Omega; M^{d\times N}\big),\ \mathcal{H}^{N-1}(J_u) < +\infty\big\} \tag{2.1}$$

and, as usual, $SBV_\infty(\Omega) := SBV_\infty(\Omega; \mathbb{R})$ whenever $d = 1$.

2.2. Piecewise constant functions and Caccioppoli partitions

Let Ω be an open set in $\mathbb{R}^N$, and let $E \subseteq \Omega$. We say that E has finite perimeter in Ω if $\mathbf{1}_E \in SBV(\Omega)$. The set of jumps of $\mathbf{1}_E$ is denoted by $\partial^* E$ and is called the reduced boundary of E: the derivative of $\mathbf{1}_E$ is concentrated on $\partial^* E$, and its total variation is given by $\mathcal{H}^{N-1} \llcorner \partial^* E$. The perimeter of E in Ω is given by $\mathcal{H}^{N-1}(\partial^* E)$.

We say that a partition $(E_i)_{i\in\mathbb{N}}$ of Ω is a *Caccioppoli partition* if $\sum_{i\in\mathbb{N}} \mathcal{H}^{N-1}(\partial^* E) < +\infty$. Given a rectifiable set $K \subset \Omega$, we say that a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω is subordinated to K if (up to a $\mathcal{H}^{N-1}$-negligible set) the reduced boundary $\partial^* E_i$ of E_i is contained in K for every $i \in \mathbb{N}$. We say that $\Omega \setminus K$ is *indecomposable* if the only Caccioppoli partition subordinated to K is the trivial one, i.e., $E_0 = \Omega$.

Caccioppoli partitions are naturally associated to piecewise constant functions, i.e., functions $u \in SBV(\Omega; \mathbb{R}^N)$ such that $\nabla u = 0$ a.e. on Ω. These functions are said piecewise constant in Ω because they are indeed constant on the subsets E_i of a Caccioppoli partition of Ω. More precisely (see [6, Theorem 4.23]) there exists a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω such that

$$u = \sum_{i\in\mathbb{N}} b_i \mathbf{1}_{E_i}, \tag{2.2}$$

with $b_i \neq b_j$ for $i \neq j$. Notice that if K is a rectifiable set in Ω such that $\Omega \setminus K$ is indecomposable, then a piecewise constant function u in Ω with $J_u \subseteq K$ is necessarily constant on Ω.

Eventually, we mention that compactness for a sequence of Caccioppoli partitions $(E_i^n)_{i\in\mathbb{N}}$, $n = 1, \ldots, \infty$, with uniformly bounded total perimeter, follows from the isoperimetric inequality in $\mathbb{R}^N$. This is used in Appendix A.

3. curl ∇u is a measure for $u \in SBV_\infty(\Omega)$

In this section, we show that the curl of the approximate gradient of a function u that satisfies the assumptions of the theorem is in fact a measure, estimated with $\mathcal{H}^{N-1} \llcorner J_u$.

Theorem 3.1. *Let $u \in SBV_\infty(\Omega)$. Then* curl ∇u *is a measure μ concentrated on J_u such that*

$$|\mu| \leqslant c\|\nabla u\|_\infty \mathcal{H}^{N-1} \llcorner J_u.$$

In this statement, the constant c depends on the dimension N. However, we conjecture that the optimal constant is $2\sqrt{2}$ (considering the Frobenius norm for matrices).

Remark 3.2. Clearly, if $u \in SBV_\infty(\Omega; \mathbb{R}^d)$ is a vector-valued function ($d \geqslant 2$), then the result still holds (with the same constant c if the norm on tensors is still the Euclidean norm of the associated matrix).

Proof. Let $u \in SBV_\infty(\Omega)$. We have:

$$u \in L^1(\Omega), \qquad L := \|\nabla u\|_\infty < +\infty, \quad \text{and} \quad \mathcal{H}^{N-1}(J_u) < +\infty.$$

The distribution curl ∇u is formally equal to the matrix $(\partial_i(\partial_j u) - \partial_j(\partial_i u))_{1 \leqslant i,j \leqslant N}$ and is defined by

$$\langle \operatorname{curl} \nabla u, \varphi \rangle = \sum_{i,j=1}^{N} \int_\Omega \partial_i u(x) \partial_j (\varphi_{i,j} - \varphi_{j,i})(x)\, dx,$$

for any $\varphi \in C_c^\infty(\Omega; M^{N\times N})$. Note that, here and in the forthcoming formulas, $\partial_i u$ denotes the ith component of the approximate gradient ∇u, which does not coincide, in general, with the ith derivative in the sense of distributions. The thesis of the theorem is local, so that it is enough to prove that if $Q \Subset \Omega$ is a hypercube in Ω, then for any $\varphi \in C_c^\infty(Q; M^{N\times N})$, one has

$$\sum_{i,j=1}^{N} \int_Q \partial_i u(x) \partial_j (\varphi_{i,j} - \varphi_{j,i})(x)\, dx \leqslant c\|\nabla u\|_\infty \|\varphi\|_\infty \mathcal{H}^{N-1}(J_u \cap Q). \tag{3.1}$$

Without loss of generality, we may assume that $Q = (0,1)^N$. We will approximate u in Q with a piecewise smooth function, jumping only on facets of smaller hypercubes. This will be done using a simplified variant of the discretization/reinterpolation technique presented in [9,10], and inspired from [16].

Step 1. Consider the set $J = J_u \cap Q$. Denote the canonical basis of $\mathbb{R}^N$ by $(e_i)_{i=1}^N$ ($e_i = (\delta_{i,j})_{j=1}^N$). One easily shows that for any i, the set

$$J_i^\varepsilon := \{-te_i + x \colon t \in [0,\varepsilon],\ x \in J\}$$

is Lebesgue-measurable. Indeed, up to a $\mathcal{H}^{N-1}$-negligible set $\mathcal{N}$, J is a countable union of compact sets: hence J_i^ε is the union of $[-\varepsilon e_i, 0] + \mathcal{N}$, which has Lebesgue measure zero, and of a countable union of compact sets. We have the estimate

$$\left|J_i^\varepsilon\right| \leqslant \varepsilon \mathcal{H}^{N-1}(J),$$

which can be derived in several ways (e.g., using the area formula), and more precisely one can show

$$\left|J_i^\varepsilon\right| \leqslant \varepsilon \int\limits_J \left|\nu_i(x)\right| d\mathcal{H}^{N-1}, \tag{3.2}$$

where $\nu(x) = (\nu_1(x), \dots, \nu_N(x))$ is the normal to J at x, defined for $\mathcal{H}^{N-1}$-a.a. $x \in J$. For $y \in (0,1)^N$, we now also define the discrete binary variable $l_{\varepsilon,i}^y(k) := \mathbf{1}_{J_i^\varepsilon}(\varepsilon y + k)$, for any $k \in \varepsilon\mathbb{Z}^N \cap Q$.

One shows that for any $i = 1, \dots, N$

$$\int\limits_{(0,1)^N} \varepsilon^{N-1} \sum_{k \in \varepsilon\mathbb{Z}^N \cap Q} l_{\varepsilon,i}^y(k)\, dy = \varepsilon^{-1} \int\limits_{(0,\varepsilon)^n} \sum_{k \in \varepsilon\mathbb{Z}^N \cap Q} \mathbf{1}_{J_i^\varepsilon}(y+k)\, dy = \varepsilon^{-1}\left|J_i^\varepsilon\right|.$$

Hence using (3.2) and $\sum_{i=1}^N |\nu_i| \leqslant \sqrt{N}$,

$$\int\limits_{(0,1)^N} \varepsilon^{N-1} \sum_{i=1}^N \sum_{k \in \varepsilon\mathbb{Z}^N \cap Q} l_{\varepsilon,i}^y(k)\, dy \leqslant \sqrt{N}\mathcal{H}^{N-1}(J).$$

Using Fatou's lemma, we deduce

$$\int\limits_{(0,1)^N} \left(\liminf_{\varepsilon \to 0} \varepsilon^{N-1} \sum_{i=1}^N \sum_{k \in \varepsilon\mathbb{Z}^N \cap Q} l_{\varepsilon,i}^y(k) \right) dy \leqslant \sqrt{N}\mathcal{H}^{N-1}(J),$$

so that for any $\delta > 0$, there exists a set A of positive measure in $(0,1)^N$, such that

$$y \in A \quad \Rightarrow \quad \liminf_{\varepsilon \to 0} \varepsilon^{N-1} \sum_{i=1}^N \sum_{k \in \varepsilon\mathbb{Z}^N \cap Q} l_{\varepsilon,i}^y(k) \leqslant \sqrt{N}\mathcal{H}^{N-1}(J) + \delta. \tag{3.3}$$

Step 2. Let now $\Delta(t) := \max\{1 - |t|, 0\}$ ($t \in \mathbb{R}$) and $\Delta_N(\xi) = \prod_{i=1}^N \Delta(\xi_i)$ for all $\xi \in \mathbb{R}^N$ (which is known in finite elements approximation as the "Q1" interpolation function). If we let

$$v_\varepsilon^y(x) := \sum_{k \in \varepsilon\mathbb{Z}^N \cap Q} u(\varepsilon y + k)\Delta_N\left(\frac{x-k}{\varepsilon} - y\right),$$

it is well known that for a.e. $y \in (0,1)^N$, $v_\varepsilon^y \to u$ in $L^1(Q)$ (see for instance [9]).

Step 3. The slicing properties of BV functions (see [6]) also ensure that for all i and $\mathcal{H}^{N-1}$-a.e. $z \in \{x \in \partial Q\colon x_i = 0\}$, the function $(0,1) \ni t \mapsto u(z + te_i)$ is in $SBV(0,1)$, with finite jump set given by $\{t\colon z + te_i \in J\}$, and whose derivative is given by $t \mapsto \partial_i u(z + te_i)$, which by assumption is bounded by L. We deduce that for a.e. $y \in (0,1)^N$, the discrete function v_ε^y satisfies $|v_\varepsilon^y(\varepsilon y + k + \varepsilon e_i) - v_\varepsilon^y(\varepsilon y + k)| \leqslant L\varepsilon$ for any $i = 1, \dots, N$ and $k \in \varepsilon\mathbb{Z}^N$ such that $\varepsilon y + k \in Q$, $\varepsilon y + k + \varepsilon e_i \in Q$ and $J \cap [\varepsilon y + k, \varepsilon y + k + \varepsilon e_i] = \emptyset$, which is equivalent to $l_{\varepsilon,i}^y(k) = 0$.

Step 4. From Steps 1–3, there exists $y \in A$ such that:

$$\liminf_{\varepsilon\to 0} \varepsilon^{N-1} \sum_{i=1}^N \sum_{k\in\varepsilon\mathbb{Z}^N\cap Q} l_{\varepsilon,i}^y(k) \leqslant \sqrt{N}\mathcal{H}^{N-1}(J) + \delta, \tag{3.4}$$

$v_\varepsilon^y \to u$ in $L^1(Q)$, and $|v_\varepsilon^y(\varepsilon y + k + \varepsilon e_i) - v_\varepsilon^y(\varepsilon y + k)| \leqslant L\varepsilon$ for any $i = 1, \dots, N$ and $k \in \varepsilon\mathbb{Z}^N$ such that $\varepsilon y + k, \varepsilon y + k + \varepsilon e_i \in Q$ and $l_{\varepsilon,i}^y(k) = 0$. We choose a sequence $(\varepsilon_j)_{j\geqslant 1}$ such that the lim inf in (3.4) is in fact a limit, and let $v_j := v_{\varepsilon_j}^y$, and $l_{j,i} := l_{\varepsilon_j,i}^y$.

From now on, since we refer only to the grids $\{\varepsilon_j y + \varepsilon_j k\colon k \in \mathbb{Z}^N\}$ which we use to interpolate u, we can assume (up to translation) that $y = 0$, so that they coincide with the grids $\{\varepsilon_j k\colon k \in \mathbb{Z}^N\}$.

In a small cube $k + (0, \varepsilon_j)^N$ in Q ($k \in \varepsilon_j\mathbb{Z}^N$), as soon as J does not intersect any edge of the cube, one has $|\partial_i v_j| \leqslant L$ for all $i = 1, \dots, N$ so that $|\nabla v_j| \leqslant \sqrt{N}L$ inside the cube. Given an edge $[k, k + \varepsilon_j e_i]$, if $l_{j,i}(k) = 1$, then J intersects the edge. In this case, we cannot control $|\nabla v_j|$ in all the cubes in Q that share this edge, whose total number is at most 2^{N-1}. We let K_j be the union of all such cubes: by (3.4) we have the estimate

$$|K_j| \leqslant 2^{N-1}\varepsilon_j^N \sum_{i=1}^N \sum_{k\in\varepsilon_j\mathbb{Z}^N\cap Q} l_{j,i}(k) \leqslant c\varepsilon_j. \tag{3.5}$$

On the other hand, we have

$$\mathcal{H}^{N-1}(\partial K_j) \leqslant (N+1)2^{N-1}\varepsilon_j^{N-1} \sum_{i=1}^N \sum_{k\in\varepsilon_j\mathbb{Z}^N\cap Q} l_{j,i}(k),$$

so that (using (3.4), with the "$\liminf_{\varepsilon\to 0}$" replaced with "$\lim_{j\to\infty}$")

$$\limsup_{j\to\infty} \mathcal{H}^{N-1}(\partial K_j) \leqslant C(N)(N+1)2^{N-1}\bigl(\sqrt{N}\mathcal{H}^{N-1}(J) + \delta\bigr). \tag{3.6}$$

Let $v_j' = v_j \mathbf{1}_{Q\setminus K_j}$. By (3.5), we still have $v_j' \to u$ in $L^1(Q)$ as $j \to \infty$. The previous discussion shows that in any $Q' \Subset Q$, for j large enough, $v_j' \in SBV(Q')$ with $\|\nabla v_j'\|_\infty \leqslant \sqrt{N}L$, v_j' is piecewise smooth and $J_{v_j'} \subseteq \partial K_j$ is a subset of a finite number of facets of hypercubes.

By Ambrosio's theorem (see [2–4] or [6, Theorem 4.36]), we know that $\nabla v_j' \rightharpoonup \nabla u$ in $L^p(Q')$ (for any $p < +\infty$). Hence $\operatorname{curl} \nabla v_j' \overset{*}{\rightharpoonup} \operatorname{curl} \nabla u$ as $j \to \infty$, in the distributional sense. On the other hand, since

$$Dv_j' = \nabla v_j'(x)\,dx + v_j'\nu_{K_j}\mathcal{H}^{N-1} \llcorner \partial K_j$$

(where ν_{K_j} is the exterior normal to K_j and v'_j stands here for the non-zero trace of v'_j on the exterior surface of K_j), and since $\operatorname{curl} Dv'_j = 0$, one has

$$\operatorname{curl} \nabla v'_j = -\operatorname{curl}\bigl(v'_j \nu_{K_j} \mathcal{H}^{N-1} \llcorner \partial K_j\bigr),$$

which can be shown to be equal to

$$-\bigl(\nabla_\tau v'_j\bigr) \wedge \nu_{K_j} \mathcal{H}^{N-1} \llcorner \partial K_j,$$

where $a \wedge b$ denotes the antisymmetric tensor product $a \otimes b - b \otimes a$. Hence its total variation, as a measure, is bounded by $\sqrt{2N} L \mathcal{H}^{N-1}(\partial K_j)$. If $\varphi \in C_c^\infty(Q; M^{N\times N})$ is fixed, one has therefore (choosing Q' such that $\operatorname{supp} \varphi \Subset Q'$),

$$\bigl\langle \operatorname{curl} \nabla v'_j, \varphi \bigr\rangle \leqslant \sqrt{2N} L \, \|\varphi\|_\infty \mathcal{H}^{N-1}(\partial K_j).$$

Passing to the limit and recalling (3.6), we get

$$\langle \operatorname{curl} \nabla u, \varphi \rangle \leqslant \sqrt{2N} L \, \|\varphi\|_\infty (N+1) 2^{N-1} \bigl(\sqrt{N} \mathcal{H}^{N-1}(J) + \delta\bigr).$$

Sending δ to zero and recalling $J = J_u \cap Q$ and $L = \|\nabla u\|_\infty$, we conclude that (3.1) holds with a constant $c \leqslant \sqrt{2} N (N+1) 2^{N-1}$. This shows the thesis of the theorem. □

Remark 3.3. The set J_u is rectifiable: for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$, if $\rho > 0$ is small enough, $J_u \cap B(x,\rho)$ is essentially a C^1 hypersurface that cuts the ball $B = B(x,\rho)$ into two disjoint Lipschitz sets, up to a set of $\mathcal{H}^{N-1}$ measure $o(\rho^{N-1})$. Moreover, up to a change of basis, we have $\nu \simeq e_1$ (and $|\nu_1| \simeq 1$, $|\nu_i| \ll 1$ for $i \geqslant 2$) in $J_u \cap B$. A similar study (see again [9,10]) will show that in such a ball B, $|\operatorname{curl} \nabla u|(B) \lesssim 2\sqrt{2N} \|\nabla u\|_\infty \mathcal{H}^{N-1}(J_u \cap B)$. Passing to the limit $\rho \to 0$, we improve the constant c in the theorem: $c \leqslant 2\sqrt{2N}$. We expect, however, that a different approximation technique, possibly not based on a discretization, would help remove the $\sqrt{N}$ in that constant.

Remark 3.4. Notice that the assumption $u \in SBV_\infty(\Omega)$ is essential in order to obtain that $\operatorname{curl} \nabla u$ is a measure absolutely continuous with respect to $\mathcal{H}^{N-1} \llcorner J_u$. In general, $\operatorname{curl} \nabla u$ is not even a measure in Ω for $u \in SBV(\Omega)$. In fact it suffices to consider $f \in L^1(\Omega)$ such that $\operatorname{curl} f$ is a distribution of order one in Ω, and the function $u \in SBV(\Omega)$ given by Alberti's result [1] such that $\nabla u = f$. More explicit counterexamples can be constructed as follows. We consider functions defined on $\Omega \subseteq \mathbb{R}^2$, so that we can identify $\operatorname{curl} \nabla u$ with the distribution

$$\langle \operatorname{curl} \nabla u, \varphi \rangle := \int_\Omega (\partial_2 u \partial_1 \varphi - \partial_1 u \partial_2 \varphi) \, dx,$$

where $\varphi \in C_c^\infty(\Omega)$.

(a) If we drop the assumption $\nabla u \in L^\infty(\Omega, \mathbb{R}^N)$, we can consider Ω as the square $Q_1 =]-1,1[^2$ of $\mathbb{R}^2$ and $u \in SBV(Q_1)$ defined as

$$u(x,y) := \begin{cases} \ln(x^2+y^2) & \text{if } y>0, \\ 0 & \text{if } y<0. \end{cases}$$

It can be easily checked that $\operatorname{curl} \nabla u$ is a distribution of order one.

(b) If we drop the assumption $\mathcal{H}^{N-1}(J_u) < +\infty$, we can reason as follows. Let ϑ be the 2-periodic function on $\mathbb{R}$ such that $\vartheta(x) = 1-|x|$ for $x \in [-1,1]$, and let $\vartheta_k(x) := \frac{1}{k}\vartheta(kx)$. Let $Q_1 =]-1,1[^2$, and let for $n \geqslant 1$

$$S_n := \left\{(x,y) \in Q_1 \colon \frac{1}{n+1} < y < \frac{1}{n}\right\}.$$

We can find $k_n \in \mathbb{N}$ in such a way that $k_n \nearrow +\infty$ and

$$u(x,y) := \begin{cases} \vartheta_{2^{k_n}}(x) & \text{if } (x,y) \in S_n, \\ 0 & \text{if } y<0 \end{cases}$$

belongs to $SBV(Q_1)$. Moreover, $|\nabla u(x,y)| = 1$ a.e. on Q_1, so that $\nabla u \in L^\infty(Q_1, \mathbb{R}^2)$. Clearly $\operatorname{curl} \nabla u$ is a Radon measure on every open set $A_n := \{(x,y) \in Q_1 \colon -\frac{1}{2} < x < \frac{1}{2}, \frac{1}{n+1} < y < \frac{3}{4}\}$ (which is compactly contained in Q_1), but $|\operatorname{curl} \nabla u|(A_n) = n-1$. As a consequence $\operatorname{curl} \nabla u$ cannot be a measure on Q_1.

4. An estimate for vector fields in a cube

This section is devoted to the proof of how we can estimate a divergence free vector field on a cube with zero normal trace in terms of the total variation of its curl: this estimate will be used in the proof of the rigidity result in order to measure how far ∇u is from a “real” gradient.

Proposition 4.1 *(Sobolev estimate for vector fields with curl measure). Let $Q = (0,1)^N$ be the unit cube in $\mathbb{R}^N$. Let $\mu \in \mathcal{M}(Q; M^{N\times N})$ be a bounded Radon measure on Q and $\varphi \in L^1(Q, \mathbb{R}^N)$ be a vector field such that*

$$\begin{cases} \operatorname{curl} \varphi = \mu & \text{in } Q, \\ \operatorname{div} \varphi = 0 & \text{in } Q, \\ \varphi \cdot \nu = 0 & \text{on } \partial Q, \end{cases} \tag{4.1}$$

where ν denotes the exterior normal to ∂Q (the first equality is in the distributional sense in Q, while the two last mean that $\varphi \mathbf{1}_Q$ has zero distributional divergence in $\mathbb{R}^N$). Then for every $1 < p < \frac{N}{N-1}$ we have that

$$\|\varphi\|_{L^p(Q,\mathbb{R}^N)} \leqslant C|\mu|(Q), \tag{4.2}$$

where C depends only on N and p, and $|\cdot|$ denotes the total variation.

Proof. We give a proof based on a duality argument and more classical elliptic estimates. Let us consider $\eta \in C_c^\infty(Q;\mathbb{R}^N)$, and let $g=(g_1,\dots,g_N)\in C^\infty(\overline{Q};\mathbb{R}^N)$ be the solution of the equation

$$\begin{cases} \Delta g=\eta & \text{in } Q,\\ g_i=0 & \text{on } \partial_{e_i^\perp}Q,\ i=1,\dots,N,\\ \frac{\partial g_i}{\partial \nu}=0 & \text{on } \partial_{e_j^\perp}Q,\ i,j=1,\dots,N,\ i\neq j, \end{cases} \tag{4.3}$$

where still, $\{e_i\colon i=1,\dots,N\}$ is the canonical basis of $\mathbb{R}^N$, and $\partial_{e_i^\perp}Q$ denotes the faces of ∂Q with $\nu=\pm e_i$. (Observe that (4.3) corresponds to finding g that minimizes the energy $\int_Q |\nabla g|^2 + 2\eta\cdot g$, with boundary condition $g\cdot\nu=0$ on ∂Q.) By elliptic regularity, for any $q\in\,]1,+\infty[$ we have the estimate

$$\|g\|_{W^{2,q}(Q,\mathbb{R}^N)}\leqslant C\|\eta\|_{L^q(Q,\mathbb{R}^N)}, \tag{4.4}$$

where C depends only on N and q. Estimate (4.4) would be standard in a smooth domain. It is shown in a two-dimensional square in [17, Theorem 4.3.2.4]. The general N-dimensional case can be easily proved in the following way: (1) first extend g periodically to the entire $\mathbb{R}^N$ by symmetrizations or antisymmetrizations across suitable hyperplanes; (2) use standard elliptic estimates (see [15, Theorem 9.11]) and a contradiction argument. (The smoothness of g, up to the boundary, also follows easily from this extension.) When $q>N$, (4.4) and Sobolev's embedding theorem yield

$$\|\nabla g\|_{C^0(\overline{Q},M^{N\times N})}\leqslant C\|\eta\|_{L^q(Q,\mathbb{R}^N)}. \tag{4.5}$$

We claim that for every $\varphi\in L^1(Q,\mathbb{R}^N)$ such that $\operatorname{curl}\varphi=\mu\in\mathcal{M}(Q,M^{N\times N})$, the following integration by parts holds

$$\int_Q \nabla g : d\mu = -\int_Q \bigl[\varphi\cdot\nabla(\operatorname{div} g)-\varphi\cdot\Delta g\bigr]\,dx. \tag{4.6}$$

Then, taking into account (4.1) and (4.3) we get

$$\int_Q \nabla g : d\mu = \int_Q \varphi\cdot\eta\,dx,$$

and in view of (4.5), for every $q>N$ we deduce

$$\left|\int_Q \varphi\cdot\eta\,dx\right|\leqslant C|\mu|(Q)\|\eta\|_{L^q(Q,\mathbb{R}^N)}.$$

Since η is arbitrary, we deduce that $\varphi\in L^p(Q,\mathbb{R}^N)$ where $p=\frac{q}{q-1}$ and that

$$\|\varphi\|_{L^p(Q,\mathbb{R}^N)}\leqslant C|\mu|(Q).$$

As q varies in $]N,+\infty[$, p ranges over $]1,\frac{N}{N-1}[$, and (4.2) follows.

In order to conclude the proof we have to prove claim (4.6). Assume first that φ is smooth up to the boundary: we show that

$$\int_Q \nabla g : \operatorname{curl}\varphi\,dx = -\int_Q \bigl[\varphi\cdot\nabla(\operatorname{div} g) - \varphi\cdot\Delta g\bigr]\,dx. \tag{4.7}$$

In fact we have

$$\begin{aligned}
&\int_Q \nabla g : \operatorname{curl}\varphi\,dx\\
&\quad=\sum_{i,j}\int_Q (\partial_i\varphi_j - \partial_j\varphi_i)\partial_j g_i\,dx\\
&\quad=\sum_{i,j}\int_{\partial Q} (\varphi_j\partial_j g_i\nu_i - \varphi_i\partial_j g_i\nu_j)\,d\mathcal{H}^{N-1} + \sum_{i,j}\int_Q \bigl(-\varphi_j\partial^2_{i,j}g_i + \varphi_i\partial^2_{j,j}g_i\bigr)\,dx.
\end{aligned}\tag{4.8}$$

Since

$$\sum_{i,j}\int_Q \bigl(-\varphi_j\partial^2_{i,j}g_i + \varphi_i\partial^2_{j,j}g_i\bigr)\,dx = -\int_Q \bigl[\varphi\cdot\nabla(\operatorname{div} g) - \varphi\cdot\Delta g\bigr]\,dx,$$

Eq. (4.7) follows if we prove that the surface terms in (4.8) vanish: but this clearly follows from the regularity of g and the boundary conditions in (4.3).

Consider now a general φ. We extend it to zero outside of Q. For each $n \geqslant 1$, we introduce the homothety of $\mathbb{R}^N$

$$x \mapsto T_n(x) := x_{1/2} + \left(1 + \frac{2}{n}\right)(x - x_{1/2}),$$

where $x_{1/2} = (\frac{1}{2}, \dots, \frac{1}{2})$ is the center of Q. If ρ is a radially symmetric smoothing kernel with support in the unit ball of $\mathbb{R}^N$ and $\rho_\varepsilon(x) = (1/\varepsilon)^N\rho(x/\varepsilon)$, we can find $\varepsilon_n < 1/n$ in such a way that setting

$$\varphi_n := \rho_{\varepsilon_n} * \varphi\bigl(T_n^{-1}(\cdot)\bigr),$$

we have $\varphi_n \to \varphi$ strongly in $L^1(\mathbb{R}^N, \mathbb{R}^N)$. Clearly, $\mu_n := (\operatorname{curl}\varphi_n)|_Q \overset{*}{\rightharpoonup} \mu$ weakly as measures on Q, and one can show that this sequence is tight: $|\mu_n|(Q \setminus K)$ can be made uniformly small if the compact set $K \Subset Q$ is well chosen (because $|\mu|(Q \setminus K)$ is arbitrarily small). From this we get that $\int_Q f\,d\mu_n \to \int_Q f\,d\mu$ for any bounded, continuous test function f, as $n \to \infty$.

Applying (4.7) to the restriction of φ_n to Q we deduce that

$$\int_Q \nabla g : d\mu_n = \int_Q \nabla g : \operatorname{curl}\varphi_n\,dx = -\int_Q \bigl[\varphi_n\cdot\nabla(\operatorname{div} g) - \varphi_n\cdot\Delta g\bigr]\,dx.$$

Letting $n \to +\infty$, we obtain (4.6). □

5. Proof of Theorem 1.2

Let us first deduce from the results in the two previous sections the following rigidity estimate.

Proposition 5.1 *(The rigidity estimate). Let* $\mathcal{K} \subset M^{N\times N}$ *be a compact set such that the quantitative rigidity estimate* (1.2) *holds for some* $p \in (1, N/(N-1))$. *Let* $Q = (0,1)^N$ *be the unit cube in* $\mathbb{R}^N$. *Let* $u \in SBV_\infty(Q;\mathbb{R}^N)$ *(cf.* (2.1)) *be such that* $\nabla u(x) \in \mathcal{K}$ *for a.e.* $x \in Q$. *Then* $\mu_u := \operatorname{curl} \nabla u$ *is a measure concentrated on* J_u *and there exists* $K \in \mathcal{K}$ *such that*

$$\|\nabla u - K\|_{L^p(Q)} \leqslant C|\mu_u|(Q), \tag{5.1}$$

where C *depends only on* N *and* p.

Proof. By Theorem 3.1 we have that $\mu_u := \operatorname{curl} \nabla u$ is a measure concentrated on J_u such that

$$|\mu_u| \leqslant c\mathcal{H}^{N-1} \llcorner J_u,$$

where c is a constant depending only on $\|\nabla u\|_\infty$ (and on N).

Let us consider $w \in H^1(Q;\mathbb{R}^N)$ solution of the minimization problem

$$\min\left\{\|\nabla v - \nabla u\|^2_{L^2(Q)}\colon v \in H^1\big(Q;\mathbb{R}^N\big),\ \int\limits_Q v(x)\,dx = 0\right\}.$$

Let $\varphi := \nabla u - \nabla w$. We have that $\varphi \in L^2(Q; M^{N\times N})$, and by minimality, that $\int_Q \varphi : \nabla v\,dx = 0$ for any $v \in H^1(Q;\mathbb{R}^N)$, hence:

$$\begin{cases} \operatorname{div}\varphi = 0 & \text{in } Q,\\ \varphi\cdot\nu = 0 & \text{on } \partial Q. \end{cases}$$

Moreover, we have that

$$\operatorname{curl}\varphi = \operatorname{curl}\nabla u - \operatorname{curl}\nabla w = \mu_u,$$

i.e., $\operatorname{curl}\varphi \in \mathcal{M}(Q; M^{N\times N})$.

By Proposition 4.1, there exists a constant C depending only on p and N such that

$$\|\varphi\|_{L^p(Q)} \leqslant C|\mu_u|(Q)$$

so that

$$\|\nabla u - \nabla w\|_{L^p(Q)} \leqslant C|\mu_u|(Q). \tag{5.2}$$

Moreover, by the rigidity estimate (1.2) we have that there exists $K \in \mathcal{K}$ such that

$$\|\nabla w - K\|_{L^p(Q)} \leqslant C\big\|\operatorname{dist}(\nabla w, \mathcal{K})\big\|_{L^p(Q)} \tag{5.3}$$

(possibly changing C, which still depends only on p and N). In view of (5.2) and (5.3), and since $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in Q$, we deduce that

$$\begin{aligned}\|\nabla u - K\|_{L^p(Q)} &\leqslant \|\nabla w - K\|_{L^p(Q)} + \|\nabla u - \nabla w\|_{L^p(Q)}\\ &\leqslant C\big\|\mathrm{dist}(\nabla w, \mathcal{K})\big\|_{L^p(Q)} + \|\nabla u - \nabla w\|_{L^p(Q)}\\ &\leqslant C\big\|\mathrm{dist}(\nabla u, \mathcal{K})\big\|_{L^p(Q)} + (1+C)\|\nabla u - \nabla w\|_{L^p(Q)}\\ &\leqslant (1+C)C|\mu_u|(Q)\end{aligned}$$

so that (5.1) holds. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Since $\nabla u(x) \in \mathcal{K}$ for a.e. $x \in \Omega$, by Theorem 3.1 we have that $\mu_u := \operatorname{curl} \nabla u$ is a measure concentrated on J_u and such that

$$|\mu_u| \leqslant c\mathcal{H}^{N-1} \llcorner J_u, \tag{5.4}$$

where $c = c(\|\nabla u\|_\infty, N)$. Let us cover $\mathbb{R}^N$ by means of disjoint cubes of side h, and let $\{Q(a_i, h)\}_{i\in I}$ be the family of these cubes contained in Ω. We can assume that $\mathcal{H}^{N-1}(J_u \cap \bigcup_i \partial Q_i) = 0$. We carry out the proof in several steps.

Step 1 *(Piecewise constant approximation of* ∇u*).* By Proposition 5.1, using a rescaling argument, we have that for every $i \in I$ there exists $K_i^h \in \mathcal{K}$ such that

$$\|\nabla u - K_i^h\|_{L^p(Q(a_i,h))} \leqslant C\frac{h^{N/p}}{h^{N-1}}|\mu_u|\big(Q(a_i, h)\big), \tag{5.5}$$

where C depends only on p and N.

Let us consider the piecewise constant function ψ_h defined on Ω such that

$$\psi_h(x) := \begin{cases} K_i^h & \text{if } x \in Q(a_i, h),\\ 0 & \text{if } x \notin \bigcup_{i\in I} Q(a_i, h).\end{cases} \tag{5.6}$$

Step 2 *(Estimate for* $|D\psi_h|$*).* Let us estimate the total variation $|D\psi_h|$ of ψ_h. We consider two neighbouring cubes $Q(a_i, h)$ and $Q(a_j, h)$. By applying estimate (5.1) to the rectangle $R_{i,j}^h = \operatorname{int}(\overline{Q(a_i, h)} \cup \overline{Q(a_j, h)})$ (of size $2h$ in one direction and h in the $N-1$ other: the proof of Proposition 4.1 in that case is identical to the proof in the case of a cube, or, alternatively, can be easily deduced by an appropriate transformation of the cube), we have that there exists $K \in \mathcal{K}$ such that

$$\|\nabla u - K\|_{L^p(R_{i,j}^h)} \leqslant \tilde{C}\frac{h^{N/p}}{h^{N-1}}|\mu_u|\big(R_{i,j}^h\big), \tag{5.7}$$

where $\tilde{C}$ depends only on N and p. Then, in view of (5.5) we get that

$$\begin{aligned}
&\left|K_i^h - K_j^h\right| \\
&\quad \leqslant \left|K_i^h - K\right| + \left|K - K_j^h\right| \leqslant 2^{1-1/p}\left(\left|K_i^h - K\right|^p + \left|K - K_j^h\right|^p\right)^{1/p} \\
&\quad = 2^{1-1/p} h^{-N/p} \left\| K - \left(K_i^h \mathbf{1}_{Q(a_i,h)} + K_j^h \mathbf{1}_{Q(a_j,h)}\right)\right\|_{L^p(R_{i,j}^h)} \\
&\quad \leqslant 2^{1-1/p} h^{-N/p}\left(\|K - \nabla u\|_{L^p(R_{i,j}^h)} + \left\|\nabla u - \left(K_i^h \mathbf{1}_{Q(a_i,h)} + K_j^h \mathbf{1}_{Q(a_j,h)}\right)\right\|_{L^p(R_{i,j}^h)}\right) \\
&\quad \leqslant 2^{1-1/p} h^{-N/p}\left(\|K - \nabla u\|_{L^p(R_{i,j}^h)} + \left\|\nabla u - K_i^h\right\|_{L^p(Q(a_i,h))} + \left\|\nabla u - K_j^h\right\|_{L^p(Q(a_j,h))}\right) \\
&\quad \leqslant 2^{1-1/p} \frac{\tilde{C} + C}{h^{N-1}} |\mu_u|\left(R_{i,j}^h\right)
\end{aligned}$$

so that

$$h^{N-1}\left|K_i^h - K_j^h\right| \leqslant C|\mu_u|\left(R_{i,j}^h\right) \tag{5.8}$$

for some new constant C depending only on N and p. We conclude that the variation of $D\psi_h$ across the interface $\partial Q(a_i,h) \cap \partial Q(a_j,h)$ is estimated with the variation of the measure μ_u in the union of the two cubes $Q(a_i,h)$ and $Q(a_j,h)$ and their common interface.

Let now A, B be open and such that $\overline{B} \subseteq A \subseteq \overline{A} \subseteq \Omega$. By (5.8) we get that for h small enough

$$|D\psi_h|(B) \leqslant C|\mu_u|(A) \tag{5.9}$$

for some C depending only on N and p.

Step 3 *(∇u is piecewise constant).* Since $\mathcal{K} \subseteq M^{N\times N}$ is compact, we have that ψ_h is uniformly bounded in $L^\infty(\Omega; M^{N\times N})$. In view of (5.9), and since $|\mu_u| \leqslant \mathcal{H}^{N-1} \llcorner J_u$, we can use the compactness in BV (see [6, Theorem 3.23]) obtaining $\psi \in BV(\Omega)$ such that

$$\psi_h \to \psi \quad \text{strongly in } L^1\left(\Omega, M^{N\times N}\right)$$

and

$$|D\psi|(A) \leqslant C\mathcal{H}^{N-1}(J_u \cap A) \tag{5.10}$$

for every open set $A \subseteq \Omega$.

Let us check that $\psi = \nabla u$. Since ∇u and ψ_h are uniformly bounded in $L^\infty(\Omega; M^{N\times N})$, and since $p < \frac{N}{N-1}$, by (5.5) we have that

$$\begin{aligned}
\limsup_{h\to 0} \|\nabla u - \psi_h\|_{L^p(\Omega)} &\leqslant \limsup_{h\to 0} \sum_{i\in I} \|\nabla u - \psi_h\|_{L^p(Q(a_i,h))} \\
&\leqslant \limsup_{h\to 0} \sum_{i\in I} C\frac{h^{N/p}}{h^{N-1}} |\mu_u|\left(Q(a_i,h)\right) \\
&\leqslant \limsup_{h\to 0} C\frac{h^{N/p}}{h^{N-1}} |\mu_u|(\Omega) = 0
\end{aligned}$$

so that $\psi_h \to \nabla u$ strongly in $L^p(\Omega; M^{N\times N})$, and $\psi = \nabla u$.

By (5.10) we get that $\nabla u \in SBV(\Omega; M^{N\times N})$, and that $D(\nabla u)$ is concentrated on J_u. Since $\mathcal{H}^{N-1}(J_u) < +\infty$, by [6, Theorem 4.23] we deduce that ∇u is piecewise constant, i.e. there exists a Caccioppoli partition $\{D_j\}_{j\in\mathbb{N}}$ and matrices $K_j \in \mathcal{K}$ such that

$$\partial^* D_j \subseteq J_u, \quad \sum_{j\in\mathbb{N}} \mathcal{H}^{N-1}(\partial^* D_j) = 2\mathcal{H}^{N-1}(J_{\nabla u}) \leqslant 2\mathcal{H}^{N-1}(J_u) \tag{5.11}$$

and

$$\nabla u = \sum_{j\in\mathbb{N}} K_j \mathbf{1}_{D_j}. \tag{5.12}$$

Step 4 *(Conclusion).* Let us consider the map $w \in SBV(\Omega)$ defined by

$$w(x) := \sum_{j\in\mathbb{N}} (K_j \cdot x)\mathbf{1}_{D_j}(x).$$

Since $\nabla w = \nabla u$, and $J_w \subseteq J_u$ in view of (5.11), we deduce that $D(u-w)$ is supported by J_u. By [6, Theorem 4.23], we conclude that there exists a Caccioppoli partition $\{F_k\}_{k\in\mathbb{N}}$ of Ω, and $b_k \in \mathbb{R}^N$, such that

$$\partial^* F_k \cap \Omega \subseteq J_u, \quad \sum_{k\in\mathbb{N}} \mathcal{H}^{N-1}(\partial^* F_k \cap \Omega) \leqslant 2\mathcal{H}^{N-1}(J_u)$$

and

$$u - w = \sum_{k\in\mathbb{N}} b_k \mathbf{1}_{F_k}. \tag{5.13}$$

Considering the Caccioppoli partition $\{E_i\}_{i\in\mathbb{N}}$ determined by the intersection of the families $\{D_j\}_{j\in\mathbb{N}}$ and $\{F_k\}_{k\in\mathbb{N}}$, we deduce that there exist $K_i \in \mathcal{K}$ and $b_i \in \mathbb{R}^N$ such that

$$u = \sum_{i\in\mathbb{N}} (K_i \cdot x + b_i)\mathbf{1}_{E_i}(x).$$

Clearly we have $J_u \subseteq \bigcup_{i\in\mathbb{N}} \partial^* E_i$. In order to conclude the proof, let us show that indeed $J_u = \bigcup_{i\in\mathbb{N}} \partial^* E_i$ up to a set of $\mathcal{H}^{N-1}$ measure zero. In fact, up to relabelling the sets E_i, we can assume that $(K_i, b_i) \neq (K_j, b_j)$ for $i \neq j$. Notice that

$$\bigcup_{i\in\mathbb{N}} \partial^* E_i \setminus J_u \subseteq \bigcup_{i\neq j} A_{ij},$$

where A_{ij} are the affine spaces

$$A_{ij} := \left\{x \in \mathbb{R}^N \colon (K_i - K_j)x = b_j - b_i\right\}.$$

The conclusion follows since A_{ij}, if not empty, has codimension at least 2, so that $\mathcal{H}^{N-1}(A_{ij}) = 0$. In fact if $K_i = K_j$, then $b_i \neq b_j$ and $A_{ij} = \emptyset$. If $K_i \neq K_j$, we have

$$\operatorname{rank}(K_i - K_j) > 1$$

(otherwise a rank-1 lamination would contradict the rigidity of $\mathcal{K}$), so that $\dim A_{ij} \leqslant N - 2$. □

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Appendix A. The linear case

A natural question is whether the linear version of Theorem 1.1 holds, that is, if a displacement $u \in SBD(\Omega)$ with $e(u) = 0$ a.e. is piecewise rigid. Let us recall that $BD(\Omega)$ is the space of displacements $u \in L^1(\Omega; \mathbb{R}^N)$ such that the symmetrized distributional derivative $\mathcal{E}u = (Du + (Du)^T)/2$ is a bounded Radon measure, and $SBD(\Omega)$ the subspace of such displacements such that this measure can be decomposed as

$$\mathcal{E}(u) = e(u)\,dx + \left(u^+ - u^-\right) \odot \nu(x)\mathcal{H}^{N-1} \llcorner J_u,$$

where u^+, u^-, ν and J_u may be defined as for SBV deformations, and $a \odot b = (a \otimes b + b \otimes a)/2$ is the symmetrized tensor product. See [5,8] for details.

The condition $\mathcal{E}(u) = 0$ may be seen as a linearization of $\nabla(Rx + \varepsilon u(x)) \in \mathrm{SO}(N)$ a.e., as $\varepsilon \to 0$, where $R \in \mathrm{SO}(N)$. In other words, it expresses the fact that the gradient of u is an antisymmetric distribution. In this case, it is known that u is in fact affine, with antisymmetric gradient. In case only the absolutely continuous part $e(u)$ vanishes, and the jump has finite energy, one has:

Theorem A.1. *Let $u \in SBD(\Omega)$ such that $\mathcal{H}^{N-1}(J_u) < +\infty$ and $e(u)(x) = 0$ for a.e. $x \in \Omega$. Then, u is a piecewise rigid displacement: there exists a Caccioppoli partition $(E_i)_{i \in \mathbb{N}}$ subordinated to J_u such that for a.e. $x \in \Omega$,*

$$u(x) = \sum_{i \in \mathbb{N}} (A_i x + b_i)\mathbf{1}_{E_i}(x),$$

where A_i is an antisymmetric $N \times N$ tensor and $b_i \in \mathbb{R}^N$ (as a consequence, $J_u = \bigcup_{i \in \mathbb{N}} \partial^* E_i$ up to a set of $\mathcal{H}^{N-1}$ measure zero). In particular, if $\Omega \setminus J_u$ is indecomposable, then u is a rigid displacement (and $J_u = \emptyset$).

Proof. The proof of this result is somewhat easier than Theorem 1.1 and we just sketch it. It turns out that if $e(u) = 0$ a.e., the one-dimensional slices $s \mapsto u(z + s\xi) \cdot \xi$ in the direction $\xi \in \mathbb{S}^{N-1}$ are piecewise constant for almost all $z \in \xi^\perp$, with a finite number of jumps: indeed, their derivative is given by $(e(u)(z + s\xi)\xi) \cdot \xi$, which is 0 [5]. Thus, we may hope that an approximation argument based on slicing, discretization and reinterpolation of the function, as in [9,10] and Section 3 will provide us with very "simple" approximating functions.

Let us first follow Steps 1–4 of the proof in Section 3: it provides us with a sequence $(u_n)_{n\geqslant 1}$ (denoted by $(v'_j)_{j\geqslant 1}$ in Section 3) such that $u_n \to u$ in $L^1(Q;\mathbb{R}^N)$ (with Q a given cube in Ω), and with, still, $u_n \in SBV(Q;\mathbb{R}^N)$ and J_{u_n} made of a finite number of facets of hypercubes, with $\mathcal{H}^{N-1}(J_{u_n}) \leqslant c\mathcal{H}^{N-1}(J_u)$. Now, instead of the Lipschitz estimate, the property that $u\cdot\xi$ is constant along slices in the direction ξ (except across J_u) yields that u_n also enjoys this property, but only in the directions e_i, $i=1,\dots,N$, along which the slicing have been performed. This is not sufficient to deduce that $e(u_n)=0$ (only, the diagonal part of $e(u_n)$, computed in the basis $(e_i)_{i=1}^N$, vanishes).

To overcome this difficulty, we need to follow the ideas in the proof of [9, Theorem 1] (see also [10] for the case $N\geqslant 3$). When choosing the origin $y\in(0,1)^N$ of our discretizations, we also consider the slices of u in the directions $(e_i+e_j)_{1\leqslant i<j\leqslant N}$ and make sure, as before, that the corresponding edges $[\varepsilon y+k, \varepsilon y+k+\varepsilon(e_i+e_j)]$, $k\in\varepsilon\mathbb{Z}^N$, cross the jump set J_u a number of times which is controlled by the total surface $\mathcal{H}^{N-1}(J_u)$. As a result, the approximate functions u_n will also satisfy, out of J_{u_n}, that $(e_i+e_j)\cdot\nabla(u_n\cdot(e_i+e_j))=0$ for all $1\leqslant i<j\leqslant N$. This is enough, now, to deduce that $e(u_n)=0$ in $Q\setminus J_{u_n}$.

Hence, such a u_n is of the form $\sum_i (A_i^n x + b_i^n)\mathbf{1}_{E_i^n}(x)$. Now, since $\mathcal{H}^{N-1}(\bigcup_i \partial E_i^n) \leqslant c\mathcal{H}^{N-1}(J_u) < +\infty$, the partition $(E_i^n)_i$ may be assumed to converge, as $n\to\infty$, to a finite-perimeter partition $(E_i)_i$ of Q. It is then easy to deduce the thesis of Theorem A.1. $\square$

The essential difference with the nonlinear case is that in the latter, we cannot ensure that our approximate functions u_n satisfy exactly the constraint: hence the need for approximate and quantitative rigidity results and a more involved proof. Observe that in the same way we can give a quite simple variant of the proof of [6, Theorem 4.23] ($u\in SBV(\Omega)$ with $\nabla u=0$ a.e. yields u piecewise constant).

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