SIAM J. MATH. ANAL.
Vol. 39, No. 1, pp. 77–102

INTERACTION OF A BULK AND A SURFACE ENERGY WITH A GEOMETRICAL CONSTRAINT*

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Abstract. This study is an attempt to generalize in dimension higher than two the mathematical results in [E. Bonnetier and A. Chambolle, *SIAM J. Appl. Math.*, 62 (2002), pp. 1093–1121]. It is the study of a physical system whose equilibrium is the result of a competition between an elastic energy inside a domain and a surface tension, proportional to the perimeter of the domain. The domain is constrained to remain a subgraph. It is shown by Bonnetier and Chambolle that several phenomena appear at various scales as a result of this competition. In this paper, we focus on establishing a sound mathematical framework for this problem in a higher dimension. We also provide an approximation, based on a phase-field representation of the domain.

Key words. epitaxial growth, surface tension, phase-field approximation, diffuse interface, Γ-convergence

AMS subject classifications. 49J45, 74N20

DOI. 10.1137/060649173

1. Introduction. In this paper, we seek to extend to a higher dimension the results of Bonnetier and Chambolle in [9]. There, the authors modelize the physical system which consists of a thin film of atoms deposited on a substrate, made of a different crystal. Such systems are common in the engineering of devices such as electronic chips, which are obtained by growing epitaxial films on flat surfaces.

In such a situation, the misfit between the crystalline lattices of the substrate and the film induces strains in the film. To release the elastic energy due to these strains, the atoms of the free surface of the film may diffuse and a reorganization occurs in the film. The result of this mechanism is a competition between the surface energy of the crystal and the bulk elastic energy. The former is roughly proportional to the free surface of the crystal and therefore favors flat configurations. The bulk energy, on the contrary, is best released if oscillatory patterns develop. We refer to [9] and the former study [11] for a more complete explanation of the phenomenon and for references on "stress driven rearrangement instabilities" and epitaxial growth.

Here, we restrict our study to the mathematical model which is proposed in [9] in dimension two. We extend to a higher dimension the relaxation result (implicitly contained in Lemma 2.1 and Theorem 2.2 in [9]) and show the correctness of the phase-field approximation, extending [9, Thm. 3.1]. Observe, however, that in that paper the bulk energy is a linearized elasticity energy that involves the symmetrized gradient of the displacement. It seems that up to now, the theory of "special bounded deformation" functions [6, 8] is not developed well enough to make possible the generalization of our results to that case so that we only work with $W^{1,p}$-coercive bulk energies. Alternatively, we could have decided to impose an additional (artificial) L^∞ constraint to the displacements, in which case the extension to linearized elasticity energies would have been relatively easy (see, for instance, [16]).

*Received by the editors January 5, 2006; accepted for publication (in revised form) October 30, 2006; published electronically April 13, 2007.
http://www.siam.org/journals/sima/39-1/64917.html
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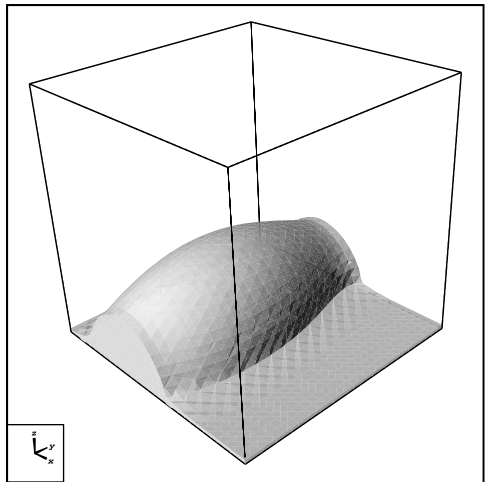

FIG. 1. *Example of an "island."*

Numerical experiments conducted by Jouve and Bonnetier [10] show that the phase-field energy introduced in section 5, in dimension three, yields results similar to the two-dimensional (2D) plots in [9]. See Figure 1 which shows how an island is formed, as a result of the competition between the surface energy and the strains in the material. Here the stretch (the lattice misfit) along the x-direction is stronger than in the y-direction, explaining the shape of the island. (In this example, the bulk energy is a linearized elasticity energy.)

To be precise, we consider in this paper a displacement in a material domain which is the subgraph of an unknown nonnegative function h. Assuming h is defined on an open Lipschitz set $\omega \subset \mathbb{R}^{N-1}$, the displacement u will be defined on the subgraph $\Omega_h := \{x = (x', x_N) \in \omega \times (0, +\infty) \,:\, x_N < h(x')\}$ of h. We will consider energies of the form

$$F(u, h) = \int_{\Omega_h} W(\nabla u)\, dx + \int_\omega \sqrt{1 + |\nabla h|^2}\, dx',$$

where u sastisfies a prescribed boundary condition on the boundary $\omega \times \{0\}$. In this paper, ω will be the $(N-1)$-dimensional torus and the boundary condition of u on "$\partial\omega$" will be of periodic type, as in [9] (however, adaption to other situations will not be difficult as long as $\partial\omega$ is Lipschitz).

The goal of our paper is to show that the relaxed functional of F can be written

$$\overline{F}(u, h) = \int_{\Omega_h} W(\nabla u)\, dx + \mathcal{H}^{N-1}(\partial_* \Omega_h) + 2\mathcal{H}^{N-1}(\Sigma),$$

where Σ, the "internal" discontinuity set of u "inside" the subgraph Ω_h of h (which is now a function of bounded variation (BV)), will be a "vertical" rectifiable set so that $\Omega_h \cup \Sigma$ can be viewed as a generalized subgraph.

In an article written almost simultaneously by Braides and the authors of the present paper [13], a similar problem is studied, without the constraint that the domain is the subgraph of a function. Although this may seem more general, showing that "recovery" sequences can be built, so that $\overline{F}$ is not only a lower bound but also an upper bound for the l.s.c. envelope of F, is considerably more difficult in our setting, since the sequence which is found must satisfy the constraint, and therefore has to be built in a constructive way (and not using some general existence result). This construction follows the discretization/reinterpolation technique introduced in [15, 16]. On the other hand, the lower bound in this work is almost a straightforward consequence of [13].

Finally, the last section in this paper deals with the phase-field approximation of $\overline{F}$, using the same approach as in [9].

2. Setting of the problem and statement of the result.

2.1. Functions of bounded variation. We start by recalling some definitions and results, useful in this paper, concerning functions of bounded variation; for this topic, we refer essentially to [7].

Let Ω be an open subset of $\mathbb{R}^N$. Given $u \in L^1(\Omega)$, its total variation is defined as

$$\sup\left\{\int_\Omega u \operatorname{div}\psi\,dx \;:\; \psi \in C_c^\infty(\Omega;\mathbb{R}^N),\ |\psi(x)| \le 1\ \forall x \in \Omega\right\}.$$

One may check that it is finite if and only if the distributional derivative Du of u is a bounded Radon measure in Ω. In this case, the total variation of u is equal to the total variation of the measure Du and is classically denoted by $|Du|(\Omega)$.

At each $x \in \Omega$, one can define upper and lower values of u as follows: The upper value is

$$u_+x(u_+(x)) \;=\; \inf\left\{t \in [-\infty,+\infty] \,:\, \limsup_{\rho\to 0}\frac{|\{y\in\Omega \,:\, u(y)>t\}| \cap B_\rho(x)}{|B_\rho(x)|} = 0\right\},$$

where $B_\rho(x)$ is the ball of radius ρ centered at x. The lower value is simply $-(-u)_+$. Defining the "jump set" of u as $S_u := \{x \in \Omega : u_-(x) < u_+(x)\}$, one can show that if $u \in BV(\Omega)$, S_u is a $(\mathcal{H}^{N-1}, N-1)$-rectifiable set (in the sense of Federer [19]) so that it admits a normal $\nu_u(x)$ at $\mathcal{H}^{N-1}$-a.e. $x \in S_u$, and Du decomposes as

$$Du \;=\; \nabla u(x)\,dx \;+\; (u_+(x) - u_-(x))\nu_u(x)\,d\mathcal{H}^{N-1} \llcorner S_u(x) \;+\; D^c u,$$

where $D^c u$, the "Cantor part," is singular with respect to the Lebesgue measure and vanishes on any set with finite $(N-1)$-dimensional Hausdorff measure. The Radon–Nikodym derivative of Du with respect to the Lebesgue measure dx, denoted by $\nabla u(x)$, is a.e. the "approximate gradient" of u at x; see [7]. Of course, if $u \in W^{1,1}(\Omega)$, it coincides with the weak gradient.

Up to now, we have considered real-valued functions. If $u\colon \Omega \to \mathbb{R}^d$ is vector-valued, S_u will be the union of the jump sets of the d components of u. One shows, then, that when two of these jump sets intersect, the corresponding normals coincide $\mathcal{H}^{N-1}$-everywhere in the intersection up to a change of sign. The jump part of the derivative Du is given by $(u_+ - u_-)\otimes\nu_u\,d\mathcal{H}^{N-1} \llcorner S_u$, where now u_+ and u_- are not the "upper" and "lower" values (since there is no natural order in $\mathbb{R}^d$) but the orientation depends on the choice of the direction of the normal ν_u (the triple (u_-, u_+, ν_u) being equivalent to $(u_+, u_-, -\nu_u)$).

The space $SBV(\Omega)$ is defined as the subset of $BV(\Omega)$ of functions u such that $D^c u = 0$, that is, Du is absolutely continuous with respect to $dx + \mathcal{H}^{N-1} \llcorner S_u$. Then, for $p > 1$, we say that a function $u\colon \Omega \to \mathbb{R}$ belongs to the space $SBV_p(\Omega)$ if $u \in SBV(\Omega)$, $\nabla u \in L^p(\Omega;\mathbb{R}^N)$, and $\mathcal{H}^{N-1}(S_u) < +\infty$.

We say that a function $u \in L^1(\Omega)$ is a *generalized function of bounded variation* ($u \in GBV(\Omega)$) if $u^T := (-T) \vee u \wedge T$ belongs to $BV(\Omega)$ for every $T \geq 0$. If $u \in GBV(\Omega)$, setting $S_u = \bigcup_{T>0} S_{u^T}$, a truncation argument allows to define the traces $u_-(x)$ and $u_+(x)$ for a.e. $x \in S_u$. Defining, for $u \in GBV(\Omega)$, the Cantor part of the derivative as $|D^c u| = \sup_{T>0} |D^c u^T|$, we say that a function u in $GBV(\Omega)$ belongs to $GSBV(\Omega)$ if $|D^c u| = 0$, and moreover u in $GSBV(\Omega)$ belongs to $GSBV_p(\Omega)$ for $p > 1$ if $\nabla u \in L^p(\Omega; \mathbb{R}^N)$ and $\mathcal{H}^{N-1}(S_u) < +\infty$.

The following compactness result for SBV is proven in [3, 5] (see also [7, Thm. 4.8]).

THEOREM 2.1 (compactness in SBV). *Let $(u_n)_n \subset SBV(\Omega)$ satisfy*

$$\sup_n \left\{ \int_\Omega |\nabla u_n|^p \, dx + \mathcal{H}^{N-1}(S_{u_n}) \right\} < +\infty,$$

with u_n uniformly bounded in $L^\infty(\Omega)$. Then, there exist a subsequence $(u_{n_k})_k$ and $u \in SBV_p(\Omega)$ such that $u_{n_k} \to u$ a.e. in Ω, $\nabla u_{n_k} \rightharpoonup \nabla u$ in $L^p(\Omega; \mathbb{R}^N)$, and

$$\mathcal{H}^{N-1}(S_u) \leq \liminf_{k \to \infty} \mathcal{H}^{N-1}(S_{u_{n_k}}).$$

If u_n is bounded only in $L^1(\Omega)$, one shows easily by truncation that the results still hold, with $u \in GSBV_p(\Omega)$.

2.2. Subgraphs of finite perimeter. In this paper, to simplify, ω is the torus $(\mathbb{R}/\mathbb{Z})^{N-1}$; however, the extension of our results to the case of a Lipschitz bounded open subset of $\mathbb{R}^{N-1}$ does not raise any difficulties. A generic point $x \in \omega \times \mathbb{R}$ will be denoted by (x', x_N), $x' = (x_1, \ldots, x_{N-1}) \in \omega$, $x_N \in \mathbb{R}$. For $h : \omega \to \mathbb{R}_+$ measurable, we consider

$$\Omega_h = \{x \in \omega \times (-1, +\infty) \; : \; x_N < h(x')\} \quad \text{and}$$

$$\Omega_h^+ = \{x \in \omega \times (0, +\infty) \; : \; x_N < h(x')\} = \Omega_h \cap (\omega \times (0, +\infty)).$$

If $h \in BV(\omega; \mathbb{R}_+)$, the set Ω_h has a finite perimeter in the sense of Caccioppoli in $\omega \times (-1, +\infty)$ (that is, $|D\chi_{\Omega_h}|(\omega \times (-1, +\infty)) \leq |\omega| + |Dh|(\omega) < +\infty$, so that $\chi_{\Omega_h} \in BV(\omega \times (-1, +\infty))$). At each point $\xi \in \omega$ one can define the upper and lower values $h_+(\xi)$ and $h_-(\xi)$ as in the previous section. As before, it is known that $h_+ = h_-$ a.e. in ω and the set of points where $h_- < h_+$, called the jump set of h, is denoted by S_h. Then, if $x = (x', x_N) \in \omega \times (-1, +\infty)$, $x_N < h_-(x') \Rightarrow x \in \Omega_h^1$ (the set of points where Ω_h has Lebesgue density 1), $x_N > h_+(x') \Rightarrow x \in \Omega_h^0$ (the set of points where it has density 0), and $\partial_* \Omega_h = \omega \times (-1, +\infty) \setminus (\Omega_h^0 \cup \Omega_h^1)$, the measure-theoretical boundary is a subset of (and $\mathcal{H}^{N-1}$-a.e. equal to) $\bigcup_{\xi \in \omega} \{\xi\} \times [h_-(\xi), h_+(\xi)]$. It is known that the measure-theoretical boundary is $\mathcal{H}^{N-1}$-a.e. equal to a subset $\partial^* \Omega_h$ called the "reduced boundary" of De Giorgi, which contains only points x where the blowups $(\Omega_h - x)/\rho$ converge as $\rho \to 0$ (in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$) to a half-space of outer normal $\nu_{\Omega_h}(x)$ (hence, Ω_h has density exactly $1/2$ at x).

Let us emphasize the fact that the boundaries $\partial \Omega_f$, $\partial_* \Omega_h$ will always, in this paper, be intended as boundaries *inside* $\omega \times (-1, +\infty)$, that is, they do not contain $\omega \times \{-1\}$.

2.3. The relaxation result. Let $W \colon M^{d \times N} \to [0, +\infty)$, with $d \geq 1$, be a continuous and quasi-convex function satisfying a p-growth condition. Let $u^0 \in W^{1,p}(\omega \times (-1, 0); \mathbb{R}^d)$.

For $h \in C^1(\omega;[0,+\infty))$ and $u \in W^{1,p}(\Omega_h^+;\mathbb{R}^d)$, with $u = u^0$ in $\omega \times \{0\}$, we set

$$F(u,h) = \int_{\Omega_h^+} W(\nabla u)\,dx + \int_\omega \sqrt{1+|\nabla h|^2}\,dx';$$

clearly, the same definition can be given for $u \in L^1(\omega \times (0,+\infty);\mathbb{R}^d)$ such that the restriction to Ω_h^+ satisfies the previous properties. Moreover, we define $F(u,h) = +\infty$ otherwise in $L^1(\omega \times (0,+\infty);\mathbb{R}^d) \times BV(\omega;[0,+\infty))$.

It is clear that equivalently one can write that $u \in W^{1,p}(\Omega_h;\mathbb{R}^d)$, with $u = u^0$ in $\omega \times (-1,0)$.

The main result of this paper is the proof of the following relaxation result for the functional F, here written in the case $d = 1$ (for the general case, see the fourth remark in section 2.4).

THEOREM 2.2. *The l.s.c. envelope of the functional F, with respect to the $L^1(\omega \times (0,+\infty)) \times L^1(\omega)$ topology, is the functional $\overline{F}\colon L^1(\omega \times (0,+\infty)) \times L^1(\omega) \to [0,+\infty]$ defined as*

$$\overline{F}(u,h) = \begin{cases} \displaystyle\int_{\Omega_h^+} W(\nabla u)\,dx + \mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(S'_u \cap \Omega_h^1) \\ \qquad \text{if } \ h \in BV(\omega;[0,+\infty)) \ \text{ and } \ u\chi_{\Omega_h^+} \in GSBV(\omega \times (0,+\infty)), \\ +\infty \quad \text{otherwise,} \end{cases}$$

where

$$S'_u \;=\; \{(x', x_N + t) \,:\, x \in S_u\,, t \geq 0\}\,.$$

Observe that, denoting $\Sigma = S'_u \cap \Omega_h^1$, Σ is a "vertical" rectifiable set, and we will sometimes write $\Gamma = \partial_*\Omega_h \cup \Sigma$, the "generalized" interface.

The proof of Theorem 2.2 will be given by showing a lower and an upper bound, respectively, in section 3 (Proposition 3.1) and in section 4 (Proposition 4.1); the thesis of Theorem 2.2 immediately follows from these results.

2.4. Some remarks.

1. In [13], a similar result is shown with mainly two differences, which both follow from the constraint that the set where u is defined is a subgraph: In the lim inf inequality, we have to keep track of the vertical parts of the boundary (S'_u) that might not be in the jump set of u (that is, one might have $(S'_u \setminus S_u) \cap \Omega_h^1 \neq \emptyset$). In the lim sup inequality, one needs to build a recovery sequence which remains a subgraph, leading to a much more complex proof than in [13].

2. In [9], one also considers the case where the surface tension for the substrate (of boundary $\omega \times \{0\}$), σ_S, can be different from the surface tension σ_C of the crystal (of boundary $\partial\Omega_h \cap (\omega \times (0,+\infty))$ if h is smooth). In this case, two different phenomena occur, depending on the fact that $\sigma_S \leq \sigma_C$ or $\sigma_C < \sigma_S$. In the latter case, it is always energetically convenient to cover (or "wet") all the surface of the substrate with an infinitesimal layer of crystal, so that the global surface tension in the relaxed energy is σ_C. In case σ_S is less than σ_C, then parts of the substrate might remain uncovered by the crystal, and the surface energy in the relaxed functional will be given by

$$\sigma_C(\mathcal{H}^{N-1}(\partial_*\Omega_h \cap (\omega \times (0,+\infty))) \,+\, 2\mathcal{H}^{N-1}(S'_u \cap \Omega_h^1)) \\ +\, \sigma_S\mathcal{H}^{N-1}(\{x' \in \omega \,:\, h(x') = 0\})\,.$$

We do not prove this result here; we fear it would make the paper harder to read, mostly because of the notation. See also Remark 4.4.

3. Still in [9], the (2D) functional F is minimized with an additional volume constraint ($\int_\omega h\,dx = 1$). It is easy to show that the relaxed functional $\overline{F}$ does not change under this constraint—see Remark 4.2 below.

4. In what follows, we will assume that $d = 1$, u is scalar, and hence W is convex. In the vectorial case, one has to assume that W is a continuous, quasi-convex function of ∇u with growth p (that is, bounded from below and above by functions of the form $a + b|\nabla u|^p$, with $b > 0$). Then, the lower bound (Proposition 3.1) remains the same thanks to results of semicontinuity for quasi-convex integrands, due to Ambrosio [4] in the SBV case (see also [7]), and to Kristensen [23] in the general case). The proof of the upper bound (Proposition 4.1, in which W does not appear) can be written with a scalar or vectorial u without any change. Then, its generalization to the Lagrangian W follows from the continuity and p-growth assumptions, as in the scalar case.

5. In [9] and the problem mentioned in the introduction, it is not u but $u - x_1$ which is 1-periodic in the first variable. Here, to simplify, everything is written with $u \in GSBV_p(\omega \times (-1, +\infty))$; that is, u is periodic in the $(N-1)$ first directions (we recall ω is the $(N-1)$-dimensional torus). Adapting the results to extend them to the case where, for instance, $u - \alpha(x_1, 0, \dots, 0) \in GSBV_p(\omega \times (-1, +\infty))$, $\alpha > 0$, would not be difficult.

3. A lower bound for the relaxed envelope of F. In this section we obtain a lower bound for the relaxed functional $\overline{F}$ by proving the following proposition.

PROPOSITION 3.1. *For every sequence $(u_n, h_n) \in W^{1,p}(\Omega_{h_n}) \times C^1(\omega; [0, +\infty))$, with $u_n = u_0$ in $\omega \times (-1, 0)$, such that*

$$\sup_n F(u_n, h_n) < +\infty,$$

there exist $h \in BV(\omega; [0, +\infty))$ and $u \in GSBV(\omega \times (0, +\infty))$ (with $u = 0$ out of Ω_h) such that $\chi_{\Omega_{h_n}} u_n \to u$ in $L^1(\omega \times (0, +\infty))$, $h_n \to h$ in $L^1(\omega)$,

$$\int_{\Omega_h^+} |\nabla u(x)|^p\,dx \le \liminf_{n\to\infty} \int_{\Omega_{h_n}^+} |\nabla u_n(x)|^p\,dx\,, \tag{1}$$

and

$$\mathcal{H}^{N-1}(\partial_* \Omega_h) + 2\mathcal{H}^{N-1}(S_u' \cap \Omega_h^1) \le \liminf_{n\to\infty} \int_\omega \sqrt{1 + |\nabla h_n(x')|^2}\,dx'\,. \tag{2}$$

This proposition implies immediately the lower bound for the relaxed envelope of F, that is, the first part of the proof of Theorem 2.2. Indeed, we obtain in the proof that the sequence $(u_n)_n$ converges in fact weakly in the $W^{1,p}$-topology, and since the function W is l.s.c. and convex, with growth p, the functional $G(u) = \int_{\Omega_h^+} W(\nabla u)\,dx$ is weakly l.s.c. in $W^{1,p}$; then, in the same hypothesis, we get the inequality

$$\begin{aligned} &\int_{\Omega_h^+} W(\nabla u(x))\,dx + \mathcal{H}^{N-1}(\partial_* \Omega_h) + 2\mathcal{H}^{N-1}(S_u' \cap \Omega_h^1) \\ &\qquad \le \liminf_{n\to\infty} \int_{\Omega_{h_n}^+} W(\nabla u_n(x))\,dx + \int_\omega \sqrt{1 + |\nabla h_n(x')|^2}\,dx'\,. \end{aligned} \tag{3}$$

Let us consider a sequence (u_n, h_n) such that

$$\sup_{n\ge 1} F(u_n, h_n) < +\infty\,;$$

we show that, up to a subsequence, $u_n \to u$ in $L^1(\omega \times (0,+\infty))$ and $h_n \to h$ in $L^1(\omega)$, with

$$\overline{F}(u,h) \leq \liminf_{n\to\infty} F(u_n,h_n). \tag{4}$$

To prove the lower inequality, it is sufficient to consider sequences (u_n, h_n) with $h_n \in C^\infty(\omega;[0,+\infty))$ and $u_n \in W^{1,p}(\Omega^+_{h_n})$, and $u_n = u^0$ on $\omega \times \{0\}$; however, this compactness property, as well as inequality (4), will still hold if we just assume that $h_n \in W^{1,1}(\omega)$ and $u_n \in SBV_p(\omega \times (-1,+\infty))$ with $u_n = u^0$ in $\omega \times (-1,0)$, $u_n(x) = 0$ a.e. in $\{x_N > h_n(x')\}$, and $S_u \tilde{\subset} \partial_* \Omega_{h_n}$ (where $A \tilde{\subset} B$ means $\mathcal{H}^{N-1}(A \setminus B) = 0$).

Let us consider first the compactness and l.s.c. of the jump term, and for this we will use a special notion of convergence for a jump set of SBV_p functions.

3.1. Jump set convegence. The following notion of jump set convergence is introduced by Dal Maso, Francfort, and Toader [18, Def. 4.1] and [17, Def. 3.1]. It is called "σ^p-convergence." A variant, which is independent on the exponent $p > 1$, has been introduced more recently by Giacomini and Ponsiglione; see [21].

In what follows, we denote equality and inclusion up to a $\mathcal{H}^{N-1}$-negligible set by the symbols $\tilde{=}$ and $\tilde{\subset}$, respectively.

DEFINITION 3.2. *Let Ω be an open set in $\mathbb{R}^N$, and let $p \in (1,+\infty)$. We say that a sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of subsets of Ω σ^p-converges to Γ if and only if* $\sup_{n\in\mathbb{N}} \mathcal{H}^{N-1}(\Gamma_n) < +\infty$ *and*

(i) *for any sequence $(v_n)_n$ of functions in $SBV_p(\Omega)$, with $S_{v_n} \tilde{\subset} \Gamma_n$, if the subsequence v_{n_k} goes to v weakly in $SBV_p(\Omega)$ as $k \to \infty$ then $S_v \tilde{\subset} \Gamma$;*

(ii) *there exists a function $v \in SBV_p(\Omega)$ and sequence $(v_n)_n$ of functions in $SBV_p(\Omega)$ converging to v such that $S_{v_n} \tilde{\subset} \Gamma_n$ for each n and $S_v = \Gamma$.*

The following compactness theorem is proven in [18, Thm. 4.7]

THEOREM 3.3. *Every sequence $\Gamma_n \subset \Omega$, with $\mathcal{H}^{N-1}(\Gamma_n)$ uniformly bounded, has a σ^p-convergent subsequence.*

The proof of this theorem is based on the following lemma (cf. [18, Lem. 4.5]).

LEMMA 3.4. *Let $(v_i)_{i=1}^\infty$ be a sequence in $SBV_p(\Omega) \cap L^\infty(\Omega)$, and let us assume $\mathcal{H}^{N-1}(\bigcup_{i=1}^\infty S_{v_i}) < +\infty$. Then there exist real numbers $c_i > 0$ with $\sum_{i=1}^\infty c_i < +\infty$ such that $v := \sum_{i=1}^\infty c_i v_i \in SBV_p(\Omega) \cap L^\infty(\Omega)$ and $S_v \tilde{=} \bigcup_{i=1}^\infty S_{v_i}$.*

Let us mention the following variant of the proof of Theorem 3.3, still based on Lemma 3.4: Given $\Gamma \subset \Omega$, we introduce

$$X(\Gamma) \ = \ \left\{ v \in SBV_p(\Omega;[-1,1]) \ : \ S_v \tilde{\subset} \Gamma \, , \ \int_\Omega |\nabla v|^p \, dx \leq 1 \right\} .$$

Then, if $\mathcal{H}^{N-1}(\Gamma) < +\infty$, by Ambrosio's compactness theorem, Theorem 2.1, $X(\Gamma)$ is compact in $L^1_{\rm loc}(\Omega)$ (which is metrizable). If $(\Gamma_n)_n$ is a sequence of jump sets with $L = \sup_n \mathcal{H}^{N-1}(\Gamma_n) < +\infty$, then the sets $X(\Gamma_n)$ all belong to

$$X_L \ = \ \left\{ v \in SBV_p(\Omega;[-1,1]) \ : \ \mathcal{H}^{N-1}(S_v) \leq L, \ \int_\Omega |\nabla v|^p \, dx \leq 1 \right\}$$

which is also compact in $L^1_{\rm loc}(\Omega)$. Hence, a subsequence $(X(\Gamma_{n_k}))_k$ converges in the Hausdorff sense (with the Hausdorff distance in $L^1_{\rm loc}(\Omega)$ induced by a distance in $L^1_{\rm loc}(\Omega)$) to a compact $K \subset X_L$. We show that $K \subseteq X(\Gamma)$ for some Γ.

Let $(v_i)_{i=1}^\infty$ be a dense sequence in the compact set K. We first observe that, since K is convex, given any v, v' in K there exists w (given by $\theta v + (1-\theta)v'$ for an appropriate choice of θ, see, for instance, [20]) such that $S_w \tilde{=} S_v \cup S_{v'}$; hence, $\mathcal{H}^{N-1}(S_v \cup S_{v'}) \le L$. In particular, we deduce that $\mathcal{H}^{N-1}(\bigcup_{i=1}^k S_{v_i}) \le L$ for any $k \ge 1$ and, passing to the limit, that $\mathcal{H}^{N-1}(\Gamma) \le L < +\infty$, where we have let $\Gamma = \bigcup_{i=1}^\infty S_{v_i}$. Using Lemma 3.4, we deduce that there exists $v \in K$ with $\Gamma \tilde{=} S_v$. Hence Γ satisfies axiom (ii) in Definition 3.2. On the other hand, any $v \in K$ is the limit of an appropriate subsequence $v_{i(k)}$, $k \ge 1$, with $S_{v_{i(k)}} \tilde{\subset} \Gamma$, and a consequence of Ambrosio's compactness theorem is that $S_v \tilde{\subset} \Gamma$, so that axiom (i) in Definition 3.2 is also satisfied. Hence Γ_{n_k} σ^p-converges to Γ. □

We observe that an obvious consequence of Ambrosio's theorem is that, if Γ_n σ^p-converges to Γ,

$$\mathcal{H}^{N-1}(\Gamma) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}(\Gamma_n)\,. \tag{5}$$

3.2. Proof of the lower inequality. Let $\Gamma_n = \partial\Omega_{h_n} = \{x \in \omega \times (-1,+\infty) : x_N = h_n(x')\}$ be the graph of the function h_n. Up to a subsequence, we know by Theorem 3.3 that Γ_n σ^p-converges to some Γ as $n \to \infty$. Since h_n is uniformly bounded in $W^{1,1}(\omega)$, possibly extracting another subsequence, $h_n \to h$ in $L^1(\omega)$. Equivalently, the sets Ω_{h_n} converge to Ω_h in the $L^1(\omega \times (0,+\infty))$ topology for the characteristic functions.

Clearly, $\partial_*\Omega_h \subseteq \Gamma$; indeed, if we take in Definition 3.2 the sequence $v_n = \chi_{\Omega_{h_n}}$, we find that $v_n \to \chi_{\Omega_h}$, whose jump set is $\partial_*\Omega_h$.

Let us decompose Γ in the three parts $\partial_*\Omega_h$, $\Sigma = \Gamma \cap \Omega_h^1$, and $\Sigma^0 = \Gamma \cap \Omega_h^0$. The part Σ^0 is irrelevant in our study since the functions u, limits of converging subsequences of (u_n), will all vanish outside of Ω_h.

We show that Σ is vertical: That is, for any $x = (x', x_N) \in \Sigma$, $(x', x_N + t) \in \Sigma \cup (\mathbb{R}^N \setminus \Omega_h^1)$ for any $t \ge 0$. Indeed, let $v \in SBV_p(\omega \times (-1,+\infty))$ be such that $S_v \tilde{=} \Gamma$, and let v_n be a sequence weakly converging to v in $SBV_p(\omega \times (-1,+\infty))$ with $S_{v_n} \tilde{\subset} \Gamma_n$. Consider the functions $x \mapsto v_n(x', x_N - t)\chi_{\Omega_{h_n}}(x)$, with $t < 1$, extended in an appropriate way in $\omega \times (-1, -1+t)$. These functions will converge to $x \mapsto v(x', x_N - t)\chi_{\Omega_h}(x)$, showing that $(S_v + te_N) \cap \Omega_h^1 \subset \Gamma$, which shows our claim. In particular, we deduce that $\mathcal{H}^{N-1}$-a.e. in Σ, $\nu_\Sigma \cdot e_N = 0$.

By (5), we have $\mathcal{H}^{N-1}(\partial_*\Omega_h) + \mathcal{H}^{N-1}(\Sigma) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}(\Gamma_n)$. We claim that, in addition,

$$\mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(\Sigma) \le \liminf_{n\to\infty} \mathcal{H}^{N-1}(\Gamma_n).$$

This follows from [13] and the definition of σ^p-convergence. Indeed, it is a consequence of the lim inf-inequality in [13], applied to a sequence $(v_n)_{n\ge1}$ with $S_{v_n} \tilde{\subset} \Gamma_n$, weakly converging in $SBV_p(\omega \times (-1,+\infty))$ to a v such that $\Sigma \tilde{\subset} S_v$.

Let us now conclude. If $F(u_n, h_n)$ is uniformly bounded, then by integration along vertical segments we easily check that (u_n) is uniformly bounded in $L^p_{\text{loc}}(\omega \times (-1,+\infty))$. Then, it is a consequence of Ambrosio's theorem, Theorem 2.1, that there exists $u \in GSBV_p(\omega \times (-1,+\infty))$ such that $u_n(x) \to u(x)$ a.e., and $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\omega \times (-1,+\infty); \mathbb{R}^N)$ so that the inequality (1) holds. Clearly, u vanishes out of Ω_h. By point (i) in Definition 3.2, which is easily generalized to $GSBV_p$ functions (see [18, Prop. 4.6]), we have that $S_u \tilde{\subset} \Sigma \cup \partial_*\Omega_h$. In particular, since Σ is vertical, $S'_u \cap \Omega_h^1 \subset \Sigma$. We deduce (2). Clearly, the inequality (4) follows from (1) and (2). □

4. An upper bound for the relaxed envelope of *F*. We now get the upper bound for the relaxed envelope of the functional F by proving the following proposition.

PROPOSITION 4.1. *For any u, h, with $\overline{F}(u,h) < +\infty$, there exist (u_n, h_n) with $h_n \in C^1(\omega; [0,+\infty))$, $u_n \in W^{1,p}(\Omega_{h_n})$, and $u_n = u^0$ in $\omega \times (-1,0)$ such that $h_n \to h$ in $L^1(\omega)$, $u_n \chi_{\Omega^+_{h_n}} \to u\chi_{\Omega^+_h}$ in $L^1(\omega \times (0,+\infty))$,*

$$\limsup_{n\to\infty} \int_{\Omega^+_{h_n}} |\nabla u_n(x)|^p \, dx = \int_{\Omega^+_h} |\nabla u(x)|^p \, dx, \tag{6}$$

and

$$\limsup_{n\to\infty} \int_\omega \sqrt{1 + |\nabla h_n(x')|^2} \, dx' \le \mathcal{H}^{N-1}(\partial_* \Omega_h) + 2\mathcal{H}^{N-1}(S'_u \cap \Omega^1_h). \tag{7}$$

We note that the proposition completes the proof of Theorem 2.2. Indeed, if we find a sequence $(u_n)_n$ satisfying (6), we can deduce the strong convergence $\nabla u_n \chi_{\Omega^+_{h_n}} \to \nabla u \chi_{\Omega^+_h}$ in L^p; the continuity of W (together with (6) and the growth condition of W) gives the general result

$$\begin{aligned} &\limsup_{n\to\infty} \int_{\Omega^+_{h_n}} W(\nabla u_n(x)) \, dx + \int_\omega \sqrt{1 + |\nabla h_n(x')|^2} \, dx' \\ &\qquad \le \int_{\Omega^+_h} W(\nabla u(x)) \, dx + \mathcal{H}^{N-1}(\partial_* \Omega_h) + 2\mathcal{H}^{N-1}(S'_u \cap \Omega^1_h), \end{aligned} \tag{8}$$

which is the lim sup inequality for the functional F.

Remark 4.2. In case one adds in the definition of functional F a volume constraint (that is, $F(u,h) = +\infty$ if $\int_\omega h \, dx \neq V$, where $V > 0$ is a fixed volume), then it is easy to show that Proposition 4.1 still holds, with the sequence (h_n) satisfying the same volume constraint as the limit h. Indeed, given the sequence (h_n) provided by the proposition (without volume constraint), one clearly has $r_n = \int_\omega h_n \, dx / \int_\omega h \, dx \to 1$ as $n \to \infty$, and an appropriate scaling (of the form $x \mapsto (x', x_N/r_n)$) of the functions and the domain will provide new sequences (u_n, h_n) with $\int_\omega h_n \, dx = \int_\omega h \, dx$, still satisfying (6) and (7).

We first state the following lemma, which shows that any BV, nonnegative subgraph with an *essentially closed* boundary can be approximated from below by the subgraph of a smooth, nonnegative function.

LEMMA 4.3. *Let $g \in BV(\omega; \mathbb{R}_+)$, and assume $\partial_* \Omega_g$ is essentially closed, that is, $\mathcal{H}^{N-1}(\overline{\partial_* \Omega_g} \setminus \partial_* \Omega_g) = 0$. Then, for any $\varepsilon > 0$, there exists $f \in C^\infty(\omega; \mathbb{R}_+)$ such that $0 \le f \le g$ a.e. in ω, $\|f - g\|_{L^1(\omega)} \le \varepsilon$ and*

$$\left| \int_\omega \sqrt{1 + |\nabla f|^2} \, dx \; - \; \mathcal{H}^{N-1}(\partial_* \Omega_g) \right| \; \le \; \varepsilon \, .$$

Proof. Consider first the distance function $d(x) = \operatorname{dist}(x, \partial_* \Omega_g)$ in $\omega \times (-1, +\infty)$. By results on the Minkowski contents [7, 19], we have (because of our assumption of essential closedness)

$$\lim_{\varepsilon \to 0} \frac{|\{x \in \omega \times (-1,+\infty) \, : \, d(x) \le \varepsilon\}|}{2\varepsilon} \; = \; \mathcal{H}^{N-1}(\partial_* \Omega_g) \, .$$

From the BV-coarea formula (and since $|\nabla d| = 1$ a.e.),

$$\frac{|\{x \, : \, d(x) \le \varepsilon\}|}{2\varepsilon} \; = \; \frac{1}{2\varepsilon} \int_{\{d \le \varepsilon\}} |\nabla d(x)| \, dx \; = \; \frac{1}{2\varepsilon} \int_0^\varepsilon \mathcal{H}^{N-1}(\partial\{d \le s\}) \, ds \, .$$

We deduce the convergence of the average values,

$$\lim_{\varepsilon\to 0} \fint_0^\varepsilon \frac{1}{2}\mathcal{H}^{N-1}(\partial\{d \le s\})\,ds \;=\; \mathcal{H}^{N-1}(\partial_*\Omega_g)\,,$$

so that we can find a sequence $(\varepsilon_k)_{k\ge 1}$ with $\varepsilon_k \downarrow 0$ such that

$$\lim_{k\to\infty} \frac{1}{2}\mathcal{H}^{N-1}(\partial\{d \le \varepsilon_k\}) \;=\; \mathcal{H}^{N-1}(\partial_*\Omega_g) \tag{9}$$

(without loss of generality we also may assume that the boundary $\partial\{d \le \varepsilon_k\}$ is Lipschitz).

Now, the boundary of $\{d \le \varepsilon_k\}$ is the disjoint union of the boundaries of $\{x \in \omega\times(-1,+\infty) : \mathrm{dist}(x,\Omega_g) \le \varepsilon_k\}$ and $\{x \in \Omega_g : d(x) > \varepsilon_k\}$, and both of these sets converge (in $L^1(\omega\times(-1,+\infty))$) to Ω_g so that the lim inf of their perimeter is greater or equal to $\mathcal{H}^{N-1}(\partial_*\Omega_g)$. Together with (9) it shows that these perimeters go to $\mathcal{H}^{N-1}(\partial_*\Omega_g)$, in particular,

$$\lim_{k\to\infty} \mathcal{H}^{N-1}(\partial\{x \in \Omega_g \,:\, d(x) > \varepsilon_k\}) \;=\; \mathcal{H}^{N-1}(\partial_*\Omega_g)\,.$$

This set $\{x \in \Omega_g : d(x) > \varepsilon_k\}$ is the subgraph in $\omega\times(-1,+\infty)$ of a function $g_k \in BV(\omega;[-\varepsilon_k,+\infty))$, with $g_k \le g - \varepsilon_k$ a.e. in ω. We consider $g_k^+ = g_k \vee 0$: We have $0 \le g_k^+$ and

$$\mathcal{H}^{N-1}(\partial\Omega_{g_k^+}) \;\le\; \mathcal{H}^{N-1}(\partial\Omega_{g_k})$$

(since $\partial\Omega_{g_k^+}\cap(\omega\times\{0\})$ is the orthogonal projection onto $\omega\times\{0\}$ of $\partial\Omega_{g_k}\cap(\omega\times(-1,0]))$. Hence, we still have

$$\lim_{k\to\infty} \mathcal{H}^{N-1}(\partial\Omega_{g_k^+}) \;=\; \mathcal{H}^{N-1}(\partial_*\Omega_g)\,.$$

By convolution, we can build from g_k^+ a sequence of smooth functions f_k that are still nonnegative, that go to g in $L^1(\omega)$, and such that

$$\lim_{k\to\infty} \int_\omega \sqrt{1+|\nabla f_k(x')|^2}\,dx' \;=\; \mathcal{H}^{N-1}(\partial_*\Omega_g)\,.$$

Let $x' \in \omega$. By construction of g_k we have $g_k(y') \le g_+(x')$ a.e. in $\{y' \in \omega : |y'-x'| \le \varepsilon_k\}$ (where $g_+ = g$ a.e. is the precise representative defined in section 2.1). Since $g \ge 0$ a.e., we also have $g_k^+(y') \le g_+(x')$ a.e. in $\{y' \in \omega : |y'-x'| \le \varepsilon_k\}$. This shows that if for each k the size of the support of the convolution kernel is chosen small enough (for instance, of diameter less than $\varepsilon_k/2$), we also have $f_k \le g_+$. This proves Lemma 4.3. $\square$

Proof of Proposition 4.1. Let us consider, now, u and h such that $\overline{F}(u,h) < +\infty$. We divide this proof into two steps.

Step 1 (approximation of (most of) the graph). We show that we can approximate a "generalized graph" $(\partial_*\Omega_h,\Sigma)$, where $\Sigma \subset \Omega_h^1\cap(\omega\times(0,+\infty))$ is vertical in the sense that $x \in \Sigma \Rightarrow (x', x_N + t) \in \Sigma$ for any $t \ge 0$ as long as $(x', x_N + t) \in \Omega_h^1$, with the graph of a smooth function $f : \omega \to \mathbb{R}_+$, with $\Omega_f \subset \Omega_h \setminus \Sigma$ up to a small part, and with a good approximation of the total surface energy $\mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(\Sigma)$ (by the surface of the smooth graph $\int_\omega \sqrt{1+|\nabla f|^2}\,dx$).

Let us first assume that $\Sigma = \emptyset$: We claim that for any $h \in BV(\omega;\mathbb{R}_+)$ and $\varepsilon > 0$, there exists $f \in C^\infty(\omega;\mathbb{R}_+)$ such that

$$\|f - h\|_{L^1(\omega)} \;+\; \mathcal{H}^{N-1}(\partial_*\Omega_h \cap \Omega_f) \;\le\; \varepsilon \tag{10}$$

and

$$\left|\int_\omega \sqrt{1+|\nabla f(x)|^2}\,dx \;-\; \mathcal{H}^{N-1}(\partial_*\Omega_h)\right| \;\le\; \varepsilon\,. \tag{11}$$

We fix $\varepsilon > 0$. Let us consider a mollifying kernel $\rho \in C_c^\infty(\mathbb{R}^N)$, with support in the unit ball, and for any $\eta > 0$ let $\rho_\eta(x) = (1/\eta)^N\rho(x/\eta)$. For $n \ge 1$ we consider the function $w_n = \rho_{1/n} * \chi_{\Omega_h} : \omega \times \mathbb{R} \to [0,1]$. It is well known not only that $w_n \to \chi_{\Omega_h}$ strongly in L^1 but also that $\int_{\omega\times(-1,+\infty)} |\nabla w_n(x)|\,dx \;\to\; |D\chi_{\Omega_h}|(\omega \times (-1,+\infty)) = \mathcal{H}^{N-1}(\partial_*\Omega_h)$ as $n \to +\infty$.

One has, for every $x \in \Omega_h^1 \cup \partial^*\Omega_h \cup \Omega_h^0$ (hence, $\mathcal{H}^{N-1}$-a.e. $x \in \omega \times (-1,+\infty)$),

$$\lim_{n\to\infty} w_n(x) \;=\; \begin{cases} 1 & \text{if } x \in \Omega_h^1\,, \\ \frac{1}{2} & \text{if } x \in \partial^*\Omega_h\,, \\ 0 & \text{if } x \in \Omega_h^0\,. \end{cases} \tag{12}$$

The same properties are true for the sequence of (l.s.c.) functions $(\tilde{w}_n)_{n\ge1}$ defined by

$$\tilde{w}_n(x) \;=\; \begin{cases} w_n(x) & \text{if } x \in \omega \times [0,+\infty)\,, \\ 1 & \text{if } x \in \omega \times (-1,0)\,. \end{cases}$$

Indeed, using the coarea formula, one sees that

$$\begin{aligned} |D\tilde{w}_n|(\omega \times (-1,+\infty)) \;&=\; \int_0^1 \mathcal{H}^{N-1}(\partial\{\tilde{w}_n > s\})\,ds \\ &\le\; \int_0^1 \mathcal{H}^{N-1}(\partial\{w_n > s\})\,ds \;=\; \int_{\omega\times(0,+\infty)} |\nabla w_n(x)|\,dx \end{aligned}$$

since $\mathcal{H}^{N-1}(\partial\{w_n > s\} \cap (\omega \times (-1,0))) \ge \mathcal{H}^{N-1}(\{x' \in \omega : w_n(x',0) \le s\}) = \mathcal{H}^{N-1}(\partial\{\tilde{w}_n > s\} \cap (\omega \times (-1,0)))$, the second set being the projection onto $\omega \times \{0\}$ of the first one. We deduce that $\limsup_{n\to\infty} |D\tilde{w}_n|(\omega \times (-1,+\infty)) \le \mathcal{H}^{N-1}(\partial_*\Omega_h)$, but since $\tilde{w}_n \to \chi_{\Omega_h}$, it yields $\lim_{n\to\infty} |D\tilde{w}_n|(\omega \times (-1,+\infty)) = \mathcal{H}^{N-1}(\partial_*\Omega_h)$. Clearly, (12) is also true for $\tilde{w}$ since $\Omega_h^1 \supset \omega \times (-1,0)$. We drop the tilde in the sequel and just write w_n instead of $\tilde{w}_n$.

For a.e. $s \in (0,1)$, one also checks that $\lim_{n\to\infty} |\{w_n > s\}\triangle\Omega_h| = 0$, and using Fatou's lemma and the coarea formula, that for a.e. $s \in (0,1)$, $\{w_n > s\}$ is an open set such that $\liminf_{n\to\infty} \mathcal{H}^{N-1}(\partial\{w_n > s\}) = \mathcal{H}^{N-1}(\partial_*\Omega_h)$. Thus, up to a subsequence (possibly depending on s), we may assume $\lim_{n\to\infty} \mathcal{H}^{N-1}(\partial\{w_n > s\}) = \mathcal{H}^{N-1}(\partial_*\Omega_h)$. Let us consider $s^* \in (2/3, 3/4)$ and an appropriate subsequence such that this property is true, and let us consider the corresponding sequence of sets $\{x \in \omega \times (-1,+\infty) : w_n(x) > s^*\}$. We have that $\mathcal{H}^{N-1}(\partial_*\Omega_h \cap \{w_n > s^*\}) = \int_{\partial_*\Omega_h} \chi_{\{w_n > s^*\}}(x)\,d\mathcal{H}^{N-1}(x)$, and since by (12), $\chi_{\{w_n>s^*\}}(x) \to 0$ $\mathcal{H}^{N-1}$-a.e. in $\partial_*\Omega_h$, we find $\mathcal{H}^{N-1}(\partial_*\Omega_h \cap \{w_n > s^*\}) \to 0$ as $n \to \infty$. We fix n large such that

$$\begin{gathered} |\{w_n > s^*\}\triangle\Omega_h| \;+\; \mathcal{H}^{N-1}(\partial_*\Omega_h \cap \{w_n > s^*\}) \;\le\; \frac{\varepsilon}{2}\,, \\ \left|\mathcal{H}^{N-1}(\partial\{w_n > s^*\}) \;-\; \mathcal{H}^{N-1}(\partial_*\Omega_h)\right| \;\le\; \frac{\varepsilon}{2}\,. \end{gathered}$$

It is clear that there exists $g : \omega \to [0,+\infty)$ a BV function such that $\{w_n > s^*\} = \{x_N < g(x')\}$. By Lemma 4.3 applied to g, we find a smooth function $f \le g$, $f \ge 0$, satisfying both (10) and (11).

Now, assume $\Sigma \neq \emptyset$. First, possibly replacing h by $h \wedge (M-1) = \min\{h, M-1\}$, $M > 1$ large, we may assume without loss of generality that h is bounded by $M-1$. Let us then define Σ' by $\Sigma' = \bigcup_{x\in\Sigma}\{x'\}\times[x_N, M]$ and recall that by assumption $\Sigma'\cap\Omega_h^1 = \Sigma$. We may also assume without loss of generality that $\mathcal{H}^{N-1}(\Sigma'\cap(\omega\times[0,M])) < +\infty$, possibly replacing (in a preliminary step) h with $h_\delta = (h-\delta)^+$, $\delta > 0$ small, and Σ with $\Sigma_\delta = \Sigma \cap \Omega_{h_\delta}$: Indeed, one will have that $\Sigma'_\delta \cap \{h_\delta(x') \le x_N \le h_\delta(x') + \delta\} \subseteq \Sigma$ so that $\mathcal{H}^{N-1}(\Sigma'_\delta \cap (\omega \times [0,M])) \le (M/\delta)\mathcal{H}^{N-1}(\Sigma) < +\infty$. Now, let $K \subseteq \Sigma'$ be a compact set such that $\mathcal{H}^{N-1}(\Sigma' \setminus K) \le \varepsilon/10$. Observe that, if K' is defined as Σ', also $\mathcal{H}^{N-1}(\Sigma' \setminus K') \le \varepsilon/10$, and K' is compact.

Let us build the sequence of l.s.c. functions $(w_n)_{n\ge1}$ and find a level $s^* \in (2/3, 3/4)$, as previously. By (12), we have that $\chi_{\{w_n>s^*\}}$ converges to 1 in Ω_h^1, while it tends to 0 $\mathcal{H}^{N-1}$-a.e. outside. In particular, $\mathcal{H}^{N-1}(K' \cap \{w_n > s^*\}) \to \mathcal{H}^{N-1}(K' \cap \Omega_h^1)$ as $n \to \infty$, and this limit satifies $\mathcal{H}^{N-1}(\Sigma) - \varepsilon/10 \le \mathcal{H}^{N-1}(K' \cap \Omega_h^1) \le \mathcal{H}^{N-1}(\Sigma)$. We can hence choose n such that

$$|\{w_n > s^*\}\triangle\Omega_h| \;+\; \mathcal{H}^{N-1}(\partial_*\Omega_h \cap \{w_n \ge s^*\}) \;\le\; \frac{\varepsilon}{4}\,,$$

$$\left|\mathcal{H}^{N-1}(\partial\{w_n > s^*\}) \;-\; \mathcal{H}^{N-1}(\partial_*\Omega_h)\right| \;\le\; \frac{\varepsilon}{4}\,,$$

and

$$\left|\mathcal{H}^{N-1}(K' \cap \{w_n > s^*\}) \;-\; \mathcal{H}^{N-1}(\Sigma)\right| \;\le\; \frac{\varepsilon}{8}\,.$$

Observe now that since the set K' is compact, its Minkowski content $|\{\mathrm{dist}(\cdot, K') < s\}|/(2s)$ converges to $\mathcal{H}^{N-1}(K')$ as $s \to 0$ (see [7, 19]). As in the proof of Lemma 4.3, we deduce that there exists a sequence $(s_k)_{k\ge1}$ such that

$$\lim_{k\to\infty} \mathcal{H}^{N-1}(\partial\{\mathrm{dist}(\cdot, K') > s_k\}) \;=\; 2\mathcal{H}^{N-1}(K')\,. \tag{13}$$

We introduce the measures $\mu_k = \mathcal{H}^{N-1} \llcorner \partial\{\mathrm{dist}(\cdot, K') > s_k\}$. Up to a subsequence, we may assume that they converge (weakly-$*$) to a measure μ supported on K'. A consequence of the lim inf inequality in [13] is that $\mu(A) \ge 2\mathcal{H}^{N-1}(K' \cap A)$ for any open set $A \subset \omega \times (0,+\infty)$. (In this simple case, it can be shown directly by a slicing argument; see, for instance, [12, Lem. 2]). Together with (13), it shows that $\mu = 2\mathcal{H}^{N-1} \llcorner K'$. In particular, if k is large enough and provided we have chosen s^* such that $\mathcal{H}^{N-1}(K' \cap \partial\{w_n > s^*\}) = 0$ (almost any choice suits, since $\mathcal{H}^{N-1}(K' \cap (\omega \times \{0\})) = 0$—otherwise $\mathcal{H}^{N-1}(\Sigma')$ would be infinite), we have

$$\left|\mathcal{H}^{N-1}(\partial\{\mathrm{dist}(\cdot, K') > s_k\} \cap \{w_n > s^*\}) \;-\; 2\mathcal{H}^{N-1}(\Sigma)\right| \;\le\; \frac{\varepsilon}{2}\,,$$

while $|\{\mathrm{dist}(\cdot, K') \le s_k\}| \le \varepsilon/4$ and $\mathcal{H}^{N-1}(\partial\{w_n > s^*\} \cap \{\mathrm{dist}(\cdot, K') \le s_k\}) \le \varepsilon/8$.

For such values of k, the open set $\{\mathrm{dist}(\cdot, K') > s_k\} \cap \{w_n > s^*\} \cap (\omega \times (-1,+\infty))$ (with piecewise Lipschitz boundary if s_k was properly chosen) is the subgraph Ω_g of a nonnegative BV function g with $\|g - h\|_{L^1(\omega)} \le \varepsilon/2$, $\mathcal{H}^{N-1}(\partial\Omega_g \setminus \partial_*\Omega_g) = 0$,

$$\mathcal{H}^{N-1}((\partial_*\Omega_h \cup \Sigma) \cap \Omega_g) \;\le\; \frac{\varepsilon}{2},$$

and $\partial\Omega_g = (\partial\{\mathrm{dist}(\cdot,K') > s_k\} \cap \{w_n > s^*\}) \cup (\partial\{w_n > s^*\} \cap \{\mathrm{dist}(\cdot,K') > s_k\})$, so that

$$\left|\mathcal{H}^{N-1}(\partial\Omega_g) \;-\; (\mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(\Sigma))\right| \;\le\; \frac{3\varepsilon}{4}.$$

Then, invoking again Lemma 4.3, we find a smooth function $f \le g$, $f \ge 0$, with $\|f-h\|_{L^1(\omega)} \le \varepsilon$,

$$\mathcal{H}^{N-1}((\partial_*\Omega_h \cup \Sigma) \cap \Omega_f) \;\le\; \varepsilon \tag{14}$$

and

$$\left|\int_\omega \sqrt{1+|\nabla f(x)|^2}\,dx \;-\; (\mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(\Sigma))\right| \;\le\; \varepsilon\,. \tag{15}$$

Remark 4.4. We have, in addition,

$$\lim_{\varepsilon\to 0} \mathcal{H}^{N-1}(\{x' \in \omega \,:\, f_\varepsilon(x') = 0\}) \;=\; \mathcal{H}^{N-1}(\{x' \in \omega \,:\, h(x') = 0\})$$

(f_ε denoting the f obtained for a particular $\varepsilon > 0$). Indeed, for $\eta > 0$, there exist $k > 1$ such that $\mathcal{H}^{N-1}(\{h < 1/k\}) \le \mathcal{H}^{N-1}(\{h = 0\}) + \eta$ and $K \subset \omega$ with $\mathcal{H}^{N-1}(K) \le \eta$ such that $f_\varepsilon \to h$ uniformly in $\omega \setminus K$. Then, if ε is small enough, $h(x') \ge 1/k$ and $x' \notin K$ will yield $f_\varepsilon(x') \ge 1/(2k)$; hence, $\{f_\varepsilon = 0\} \subset K \cup \{h < 1/k\}$ so that $\mathcal{H}^{N-1}(\{f_\varepsilon = 0\}) \le \mathcal{H}^{N-1}(\{h = 0\}) + 2\eta$. We deduce that $\limsup_{\varepsilon\to 0} \mathcal{H}^{N-1}(\{f_\varepsilon = 0\}) \le \mathcal{H}^{N-1}(\{h = 0\})$. On the other hand, since $\mathcal{H}^{N-1}(\partial_*\Omega_h \cap \Omega_{f_\varepsilon}) \to 0$, we see that $\mathcal{H}^{N-1}(\{h = 0\} \cap \{f_\varepsilon > 0\}) \to 0$ so that $\mathcal{H}^{N-1}(\{h = 0\} \cap \{f_\varepsilon = 0\}) \to \mathcal{H}^{N-1}(\{h = 0\})$; hence, $\mathcal{H}^{N-1}(\{h = 0\}) \le \liminf_{\varepsilon\to 0} \mathcal{H}^{N-1}(\{f_\varepsilon = 0\})$.

A consequence is that in case (as in [9]) the "substrate" $\{x_N \le 0\}$ has a superficial tension σ_S less than the superficial tension σ_C of the crystal, that is, the surface energy of $(\partial_*\Omega_h, \Sigma)$ is

$$\sigma_S \mathcal{H}^{N-1}(\{h = 0\}) \;+\; \sigma_C(\mathcal{H}^{N-1}(\partial_*\Omega_h \cap (\omega \times (0,+\infty))) + 2\mathcal{H}^{N-1}(\Sigma))\,,$$

then f can fulfill the additional requirement

$$\Big|\sigma_S \mathcal{H}^{N-1}(\{f = 0\}) + \sigma_C \int_{\{f>0\}} \sqrt{1+|\nabla f|^2}\,dx \\ - \big(\sigma_S \mathcal{H}^{N-1}(\{h = 0\}) + \sigma_C(\mathcal{H}^{N-1}(\partial_*\Omega_h \cap (\omega \times (0,+\infty))) + 2\mathcal{H}^{N-1}(\Sigma))\big)\Big| \le \varepsilon.$$

If on the other hand $\sigma_C < \sigma_S$, this is not optimal (in terms of relaxation, approximating (h,Σ) with $(h+\delta, \Sigma + \delta e_N)$, δ small, will reduce the energy).

Step 2. (approximation of both the graph and the displacement). We now show that if $u \in GSBV_p(\omega \times (-1,+\infty))$ is given, with $S_u \subseteq \partial_*\Omega_h \cup \Sigma$, $u = 0$ out of Ω_h, and $u = u^0$ on $\omega \times (-1,0)$ (where $u^0 \in W^{1,p}(\omega \times (-1,0))$, $\Sigma \subset \Omega_h^1 \cap (\omega \times (0,+\infty))$ vertical), then there exists $(u_n,h_n)_{n\ge 1}$, with $h_n \in C^\infty(\omega;\mathbb{R}_+)$, $u_n \in W^{1,p}(\Omega_{h_n})$, and $u_n = u^0$ in $\omega \times (-1,0)$, such that as $n \to \infty$, $h_n \to h$ in $L^1(\omega)$, and (extending both u_n and ∇u_n with zero out of Ω_{h_n}) $u_n \to u$ in $L^1(\omega \times (-1,+\infty))$, $\nabla u_n \to \nabla u$ strongly in $L^p(\omega \times (-1,+\infty);\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_\omega \sqrt{1+|\nabla h_n(x)|^2}\,dx \;=\; \mathcal{H}^{N-1}(\partial_*\Omega_h) \;+\; 2\mathcal{H}^{N-1}(\Sigma)\,.$$

Before entering the proof of this second step, which is very technical, let us give a rough idea of how it goes. It follows a discretization/reinterpolation argument introduced in [15, 16]. We first discretize the function u in $\omega \times (-1, +\infty)$ on a regular square grid (of the form $\eta\mathbb{Z}^N$ with η small). Then, we reinterpolate this discretization into a piecewise continuous function u^η. All this is done in a way that ensures that some suitable volume energy of u^η is controlled by the same energy of u and converges in the limit $\eta \to 0$. On the other hand, the jump set of this approximation u^η is "close" in some sense to the jump set of u, but a drawback of this technique is that we can control only its total surface by $C \times (\mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(\Sigma))$ (with C a large constant depending on the dimension N).

To overcome this difficulty, we have to use the approximation f provided by the previous step: Instead of doing the construction in the whole set $\omega \times (-1, +\infty)$, we work in the smooth set Ω_f, where f is such that the jump of u in Ω_f has a surface of order ε (and satisfies (15)). In this way, the jump set of u^η in Ω_f is now controlled by $C\varepsilon$, and after extending u^η to $\omega \times (-1, +\infty)$ by zero above the graph of f, we get a couple (u^η, h^η) with total surface energy controlled by $\mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(\Sigma) + (C+1)\varepsilon$, as required.

Let us now enter the details. We fix $\varepsilon > 0$. By Step 1, there exists $f \in C^\infty(\omega)$ with $\|f - h\|_{L^1(\omega)} \le \varepsilon$, such that both (14) and (15) hold. (In particular, (14) states that f is "almost" below h (and Σ) in the sense that very little of $\partial_*\Omega_h \cup \Sigma$ lies below f.) We denote by v the $GSBV_p$ function that is equal to u in Ω_f, to 0 in $(\omega \times (0, +\infty)) \setminus \Omega_f$, and to u^0 in $\omega \times (-1, 0)$. Possibly choosing f closer to h, we may assume, also, that $\|v - u\|_{L^1(\omega\times(-1,+\infty))} \le \varepsilon$. Eventually, we also extend v (by symmetry) slightly below $\omega \times \{-1\}$ to the set $\omega \times (-1-\delta, -1)$, $0 < \delta < 1$.

Let us define, for $\xi \in \mathbb{R}^N$, the anisotropic potential

$$W_p(\xi) \ := \ \sum_{i=1}^N |\xi_i|^p \,.$$

Clearly, $v \in GSBV_p(\omega \times (-1-\delta, +\infty))$, and one has, if δ is small enough,

$$\begin{aligned}(16)\qquad \int_{\Omega_f^\delta} W_p(\nabla v(x))\,dx \ &= \ \int_{\omega\times(-1-\delta,+\infty)} W_p(\nabla v(x))\,dx \\ &\le \ \int_{\omega\times(-1,+\infty)} W_p(\nabla u(x))\,dx + \varepsilon,\end{aligned}$$

where $\Omega_f^\delta = \{x \in \omega \times (-1-\delta, +\infty) \,:\, x_N < f(x')\}$. The jump set of v satisfies $S_v \subset \partial\Omega_f \cup ((\partial_*\Omega_h \cup \Sigma) \cap \Omega_f)$; its surface energy is estimated by (14) and (15).

For $n \ge 1$, let $\eta = 1/n$ be a discretization step. Given $y \in (0,1)^N$, we introduce a discretization of v by setting $v_k^{y,\eta} = v(y\eta + k\eta)$, with $(k_1, \dots, k_{N-1}) \in (\mathbb{Z}/n\mathbb{Z})^{N-1}$ and $k_N \in \mathbb{Z} \cap [-(1+\delta)/\eta - y_N, +\infty)$ (so that only points in $\omega \times (-1-\delta, +\infty)$ are considered).

We also define a "discrete jump" of $v^{y,\eta}$: We let, for $i = 1, \dots, N$, and y, k as above, $l_k^{i,y,\eta} = 0$ if $(\partial_*\Omega_h \cup \Sigma) \cap [y\eta + k\eta, y\eta + (k+e_i)\eta] = \emptyset$ and $l_k^{i,y,\eta} = 1$ otherwise. We have that $l^{i,y,\eta} = \chi_{S_\eta^i}(y\eta + k\eta)$, where the set S_η^i is given by

$$S_\eta^i \ = \ (\partial_*\Omega_h \cup \Sigma) \ + \ [-\eta e_i, 0] \,.$$

Here $(e_1, \dots, e_N)$ is the canonical basis of $\mathbb{R}^N$, and as usual the sum of two sets A, B is $A + B = \{a + b \,:\, a \in A, b \in B\}$.

The discrete energy of $(v_k^{y,\eta}, (l_k^{i,y,\eta})_{i=1}^N)_k$ is defined by

$$D_\eta^y \;=\; \sum_{i=1}^N D_\eta^{i,y} \quad\text{with}\quad D_\eta^{i,y} = \eta^N \sum_k (1-l_k^{i,y,\eta}) \frac{|v_{k+e_i}^{y,\eta} - v_k^{y,\eta}|^p}{\eta^p} + \alpha \frac{l_k^{i,y,\eta}}{\eta}\,,$$

where the sum is taken on all k such that the segment $[y\eta + k\eta, y\eta + (k+e_i)\eta]$ lies inside the subgraph Ω_f^δ. The parameter $\alpha > 0$ will be fixed later on.

Let us compute the average $\int_{y\in(0,1)^N} D_\eta^y\,dy$. For each i, one has (using the change of variable $(y,k)\mapsto x = (y+k)\eta$)

$$\text{(17)}\qquad \int_{(0,1)^N} D_\eta^{i,y}\,dy \;=\; \int_{\mathcal{O}_\eta^i} (1-\chi_{S_\eta^i})(x)\frac{|v(x+\eta e_i) - v(x)|^p}{\eta^p} \;+\; \alpha\frac{\chi_{S_\eta^i}(x)}{\eta}\,dx\,,$$

where the domain of integration is

$$\mathcal{O}_\eta^i = \left\{ x\in\omega\times(-1-\delta,+\infty)\,:\, x_N < \min_{0\le t\le 1} f(x'+t\eta e_i)\right\} \quad \text{if } i\le N-1, \text{ and}$$
$$\mathcal{O}_\eta^N \;=\; \{x\in\omega\times(-1-\delta,+\infty)\,:\, x_N < f(x') - \eta\}\,.$$

We now use a slicing technique introduced by Gobbino [22] (based on the slicing properties of $GSBV$ functions [7] and applied in a similar setting in [2, 15, 16]). The second integral in (17) is decomposed into an integral on $e_i^\perp$ and an integral along the direction e_i, as follows:

$$\int_{(0,1)^N} D_\eta^{i,y}\,dy \;=\;$$
$$\int_{e_i^\perp} d\mathcal{H}^{N-1}(z) \int_{\{s: z+se_i\in\mathcal{O}_\eta^i\}} (1-\chi_{S_\eta^i})(z+se_i)\frac{|v(z+(s+\eta)e_i) - v(z+se_i)|^p}{\eta^p}$$
$$\;+\;\alpha\frac{\chi_{S_\eta^i}(z+se_i)}{\eta}\,ds\,.$$

For $\mathcal{H}^{N-1}$-a.e. $z\in e_i^\perp$, from the definition of S_η^i there is no jump of v between $z+se_i$ and $z+(s+\eta)e_i$ (for a.e. s) when $(1-\chi_{S_\eta^i})(z+se_i)\ne 0$, so that in this case

$$|v(z+(s+\eta)e_i) - v(z+se_i)|^p \;=\; \left|\int_0^\eta \frac{\partial v}{\partial x_i}(z+(s+t)e_i)\,dt\right|^p$$
$$\le\; \eta^{p-1}\int_0^\eta \left|\frac{\partial v}{\partial x_i}(z+(s+t)e_i)\right|^p\,dt\,.$$

We deduce

$$\text{(18)}\quad \int_{e_i^\perp} d\mathcal{H}^{N-1}(z)\int_{\{s:z+se_i\in\mathcal{O}_\eta^i\}} (1-\chi_{S_\eta^i})(z+se_i)\frac{|v(z+(s+\eta)e_i) - v(z+se_i)|^p}{\eta^p}$$
$$\le\; \int_{\Omega_f^\delta}\left|\frac{\partial v}{\partial x_i}(x)\right|^p\,dx\,.$$

On the other hand, for $\mathcal{H}^{N-1}$-a.e. z, we have (from the definition of S_η^i)

$$\left|\{s: z+se_i\in\mathcal{O}_\eta^i\cap S_\eta^i\}\right| \;\le\; \eta\mathcal{H}^0\left(\{s: z+se_i\in\mathcal{O}_\eta^i\cap(\partial_*\Omega_h\cup\Sigma)\}\right)$$

so that

$$\int_{e_i^\perp} d\mathcal{H}^{N-1}(z) \int_{\{s: z+se_i \in \mathcal{O}^i_\eta\}} \frac{\chi_{S^i_\eta}(z+se_i)}{\eta}\, ds \le \int_{(\partial_*\Omega_h\cup\Sigma)\cap\Omega_f} |e_i\cdot\nu(x)|\, d\mathcal{H}^{N-1}(x)\,, \tag{19}$$

where ν is the normal to $\partial_*\Omega_h\cup\Sigma$, defined $\mathcal{H}^{N-1}$-a.e. (up to a change of sign which is not relevant here). Collecting (18) and (19), we get

$$\int_{(0,1)^N} D^{i,y}_\eta\, dy \;\le\; \int_{\Omega^\delta_f} \left|\frac{\partial v}{\partial x_i}(x)\right|^p dx \;+\; \alpha \int_{(\partial_*\Omega_h\cup\Sigma)\cap\Omega_f} |e_i\cdot\nu(x)|\, d\mathcal{H}^{N-1}(x)\,.$$

By construction, we have $\mathcal{H}^{N-1}((\partial_*\Omega_h\cup\Sigma)\cap\Omega_f)\le\varepsilon$ (see (14)); hence,

$$D^y_\eta \;=\; \sum_{i=1}^N \int_{(0,1)^N} D^{i,y}_\eta \;\le\; \int_{\Omega^\delta_f} W_p(\nabla v(x))\, dx \;+\; \alpha\sqrt{N}\varepsilon\,. \tag{20}$$

Now, for any y and $\eta>0$ (small), we introduce the interpolate (known as "Q1" in finite elements theory) of $(v^{y,\eta}_k)_k$:

$$v^{y,\eta}(x) \;=\; \sum_{k\in(\mathbb{Z}/n\mathbb{Z})^{N-1}\times\mathbb{Z}} v^{y,\eta}_k \Delta\left(\frac{x}{\eta}-(k+y)\right)\,,\quad x\in\omega\times\mathbb{R}\,,$$

where (as before $a^+ = a\vee 0 = \max\{a,0\}$)

$$\Delta(x) \;=\; \prod_{i=1}^N (1-|x_i|)^+\,. \tag{21}$$

It is classical [2, 14] that there exists a sequence $(\eta_l)_{l\ge1}$ such that $v^{y,\eta}\to v$ in $L^1(\omega\times(-1,+\infty))$ as $l\to\infty$ for a.e. $y\in(0,1)^N$. Then, possibly extracting a subsequence, we deduce from (20) that there exists $y\in(0,1)^N$ such that both

$$\lim_{l\to\infty} D^y_{\eta_l} \;\le\; \int_{\omega\times(-1-\delta,+\infty)} W_p(\nabla v)\, dx \;+\; \alpha\sqrt{N}\varepsilon \tag{22}$$

and $\|v^{y,\eta_l}-v\|_{L^1}\to 0$ as $l\to\infty$. In what follows, we fix y to this value and drop the corresponding superscript.

Consider now a cube $C_k=(y+k)\eta_l+(0,\eta_l)^N$ such that $C_k\subset\Omega^\delta_f$. We say that C_k is a "regular cube" if $\partial_*\Omega_h\cup\Sigma$ does not cross any edge of C_k, that is, when $l^{i,\eta_l}_{\hat{k}}=0$ for any i and $\hat{k}\in k+\{0,1\}^N$ with $\hat{k}_i=k_i$. On the other hand, if $\partial_*\Omega_h\cup\Sigma$ crosses at least one of the edges of C_k, we say the cube is a "jump cube." Notice that in the latter case, since $\partial_*\Omega_h\cup\Sigma$ is a generalized subgraph, all cubes "above" C_k are also jump cubes as long as they intersect Ω_h: Precisely, every other cube $C'=C_{k',k_N+m}$, $m\ge1$, with $C'\subset\Omega^\delta_f$ has at least one edge that crosses $\partial_*\Omega_h\cup\Sigma$ unless $C'\subset\Omega_f\setminus\Omega^1_h$ (in which case $v=0$ and $v^{\eta_l}=0$ a.e. in C'). We denote by $\mathcal{J}$ the union of all jump cubes and of all cubes in Ω^δ_f that lie above a jump cube (i.e., either jump cubes or regular cubes where v vanishes) and by $\mathcal{R}$ the union of all the other regular cubes (the regular cubes that lie below $\partial_*\Omega_h\cup\Sigma$) so that $\mathcal{C}_f=\mathcal{R}\cup\mathcal{J}$ is the union of all cubes $C_k\subset\Omega^\delta_f$ (see Figure 2).

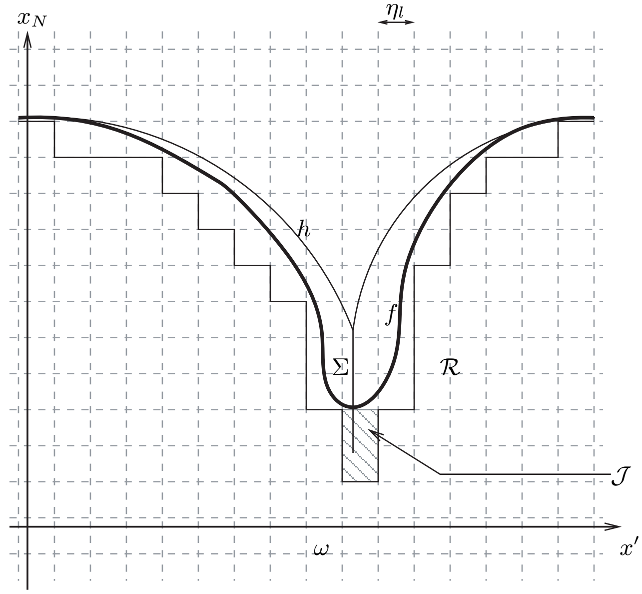

FIG. 2. *The cubes below f are grouped into two regions: a region $\mathcal{R}$ where the total bulk energy of v^{η_l} is estimated by the "bulk" part of D_{η_l} and a region $\mathcal{J}$ whose common boundary with $\mathcal{R}$ is estimated by the "surface term," or order ε, of D_{η_l}.*

Since each edge $[(y+k)\eta_l,(y+k+e_i)\eta_l]$ (for any i) is shared by at most 2^{N-1} cubes, one can decompose the energy D_{η_l} as a sum of contributions of the cubes in the following way:

$$D_{\eta_l} \;\geq\; \sum_{C_k \text{ "regular"}} (\eta_l)^N \frac{1}{2^{N-1}} \sum_{i=1}^{N} \sum_{\substack{\hat{k}\in k+\{0,1\}^N \\ \hat{k}_i=k_i}} \frac{\left|v^{\eta_l}_{\hat{k}+e_i}-v^{\eta_l}_{\hat{k}}\right|^p}{(\eta_l)^p} + \alpha\left(\frac{\eta_l}{2}\right)^{N-1} \times (\text{number of "jump cubes"})$$

(this is a very rough estimate since only one edge of each jump cube is taken into account even if many edges cross $\partial_*\Omega_h\cup\Sigma$). In particular, from (20) we find that the number of the jump cubes is bounded by a constant times η_l^{1-N}, so that their total Lebesgue measure is $O(\eta_l)$.

By inequality (36) in Lemma A.1, the term in the sum over regular cubes is larger or equal to $\int_{C_k} W_p(\nabla v^{\eta_l}(x))\,dx$; on the other hand, the term involving the jump cubes bounds the measure of the boundary of these cubes since clearly $\alpha(\eta_l/2)^{N-1} = \alpha\mathcal{H}^{N-1}(\partial C_k)/(N2^N)$. In particular, we have

$$D_{\eta_l} \;\geq\; \int_{\mathcal{R}} W_p(\nabla v^{\eta_l}(x))\,dx + \frac{\alpha}{N2^N}\mathcal{H}^{N-1}(\partial\mathcal{J}\cap\partial\mathcal{R})\,. \tag{23}$$

Now, we will "move" $\mathcal{C}_f=\mathcal{R}\cup\mathcal{J}$ upwards (in the direction x_N) in order to cover Ω_f (we will then translate v^{η_l} accordingly): Let $\kappa = 1+\sqrt{N}\max_{\xi\in\omega}|\nabla f(\xi)|$; this constant is such that

$$\mathcal{C}_f+\kappa\eta_l e_N \;\supset\; \Omega_f$$

as soon as l is large enough (so that $x_N > -1$ yields $x_N - \kappa\eta_l > -1 - \delta + \eta_l$ which clearly holds as soon as $\eta_l \le \delta/(1+\kappa)$).

We then define, for any l (large enough), the function $f_l \in BV(\omega)$ by $f_l(x') = \sup\{x_N < f(x') : (x', x_N - \kappa\eta_l) \in \mathcal{R}\}$, and for any $x \in \omega \times (-1, +\infty)$, we also define $v_l(x)$ by

$$v_l(x) \;=\; \begin{cases} v^{\eta_l}(x', x_N - \kappa\eta_l) & \text{if } -1 < x_N < f_l(x'), \\ 0 & \text{otherwise.} \end{cases}$$

By construction, the boundary of Ω_{f_l} (in $\omega \times (-1, +\infty)$) is a piecewise smooth compact set made of two parts: One part is contained in the (smooth) graph of f, $\partial\Omega_f$, and the rest, $\partial\Omega_{f_l} \cap \Omega_f$, is a subset of $(\partial\mathcal{J} \cap \partial\mathcal{R}) + \kappa\eta_l e_N$, which is a finite union of facets of hypercubes. On the other hand, $v_l \in W^{1,p}(\Omega_{f_l})$, with as a consequence of (23),

$$\int_{\Omega_{f_l}} W_p(\nabla v_l(x))\,dx \;+\; \frac{\alpha}{N2^N}\mathcal{H}^{N-1}(\partial\Omega_{f_l} \cap \Omega_f) \;\le\; D_{\eta_l}\,. \tag{24}$$

We fix $\alpha = N2^N$ and make the observation that $v_l = v^{\eta_l}(\cdot - \kappa\eta_l e_N)$ except on a set of measure $O(\eta_l)$ (the union of the cubes of $\mathcal{J}$ such that $\partial_*\Omega_h \cup \Sigma$ crosses an edge of the cube). Therefore, $v_l \to v$ as $l \to \infty$ in $L^1(\omega \times (-1, +\infty))$ (and, as well, $f_l \to f$). We can now fix l large enough so that $\|f_l - f\|_{L^1(\omega)} + \|v_l - v\|_{L^1(\omega\times(-1,+\infty))} < \varepsilon$ and

$$\begin{aligned}\int_{\Omega_{f_l}} W_p(\nabla v_l(x))\,dx \;+\; \mathcal{H}^{N-1}(\partial\Omega_{f_l}) \;&\le\; D_{\eta_l} \;+\; \mathcal{H}^{N-1}(\partial\Omega_f)\\ &\le\; \int_{\omega\times(-1,+\infty)} W_p(\nabla u(x))\,dx \;+\; \mathcal{H}^{N-1}(\partial_*\Omega_h) \;+\; 2\mathcal{H}^{N-1}(\Sigma) \;+\; (3 + 2^N N\sqrt{N})\varepsilon\,,\end{aligned}$$

where we have used (15), (16), (22), and (24). Observe eventually that if l is large enough, we also have (since $\liminf_{l\to\infty} \mathcal{H}^{N-1}(\partial\Omega_{f_l}) \ge \mathcal{H}^{N-1}(\partial\Omega_f)$ and using (15))

$$\mathcal{H}^{N-1}(\partial\Omega_{f_l}) \;\ge\; \mathcal{H}^{N-1}(\partial_*\Omega_h) \;+\; 2\mathcal{H}^{N-1}(\Sigma)) \;-\; 2\varepsilon\,.$$

Using now Lemma 4.3, we can find a smooth $f' \in C^\infty(\omega;\mathbb{R}^N)$ with $f' \le f_l$, close enough to f_l, in such a way that if $v' = v_l$ in Ω'_f and 0 in $(\omega \times (-1, +\infty)) \setminus \Omega'_f$, one has $\|f' - f\|_{L^1(\omega)} + \|v' - v\|_{L^1(\omega\times(-1,+\infty))} < 2\varepsilon$; hence, both $\|f' - h\|_{L^1(\omega)} < 3\varepsilon$ and $\|v' - u\|_{L^1(\omega\times(-1,+\infty))} < 3\varepsilon$, and

$$\begin{aligned}&\int_{\Omega_{f'}} W_p(\nabla v'))\,dx \;+\; \mathcal{H}^{N-1}(\partial\Omega_{f'})\\ &\qquad\le\; \int_{\omega\times(-1,+\infty)} W_p(\nabla u(x))\,dx \;+\; \mathcal{H}^{N-1}(\partial_*\Omega_h) \;+\; 2\mathcal{H}^{N-1}(\Sigma) \;+\; \beta\varepsilon,\end{aligned}$$

where $\beta = 4 + 2^N N\sqrt{N}$ is a constant and, as well,

$$\mathcal{H}^{N-1}(\partial\Omega_{f'}) \;\ge\; \mathcal{H}^{N-1}(\partial_*\Omega_h) \;+\; 2\mathcal{H}^{N-1}(\Sigma)) \;-\; 3\varepsilon\,.$$

Performing this construction for $\varepsilon = 1/n$, $n \ge 1$, yields the existence of two sequences $(f_n)_{n\ge1}$, $(u_n)_{n\ge1}$, with $f_n \in C^\infty(\omega)$, $u_n \in W^{1,p}(\Omega_{f_n})$, $f_n \to h$ in $L^1(\omega)$,

$u_n \to u$ in $L^1(\omega \times (-1,+\infty))$,

$$\limsup_{n\to\infty} \int_{\Omega_{f_n}} W_p(\nabla u_n(x))\,dx + \int_\omega \sqrt{1+|\nabla f_n(x)|^2}\,dx \leq \int_{\omega\times(-1,+\infty)} W_p(\nabla u(x))\,dx + \mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(\Sigma)\,, \tag{25}$$

and

$$\mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(\Sigma) \leq \liminf_{n\to\infty} \int_\omega \sqrt{1+|\nabla f_n(x)|^2}\,dx\,. \tag{26}$$

The function u_n, extended with 0 out of Ω_{f_n}, is in $GSBV(\omega\times(-1,+\infty))$, and its gradient is ∇u_n in Ω_{f_n} and 0 outside. Invoking now Ambrosio's compactness theorem for $GSBV$ functions, we find that $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\omega\times(-1,+\infty);\mathbb{R}^N)$, so that

$$\int_{\omega\times(-1,+\infty)} W_p(\nabla u(x))\,dx \leq \liminf_{n\to\infty} \int_{\omega\times(-1,+\infty)} W_p(\nabla u_n(x))\,dx\,,$$

which, combined with (25) and (26), yields that

$$\lim_{n\to\infty} \int_{\omega\times(-1,+\infty)} W_p(\nabla u_n(x))\,dx = \int_{\omega\times(-1,+\infty)} W_p(\nabla u(x))\,dx\,, \tag{27}$$

$$\lim_{n\to\infty} \int_\omega \sqrt{1+|\nabla f_n(x)|^2}\,dx = \mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(\Sigma))\,. \tag{28}$$

In particular, we deduce from (27) (since $1<p<+\infty$) that ∇u_n goes strongly to ∇u in $L^p(\omega\times(-1,+\infty);\mathbb{R}^N)$. We also find that $u_n \to u^0$ strongly in $W^{1,p}(\omega\times(-1,0))$. Modifying u_n in order to ensure that $u_n \equiv u^0$ in $\omega\times(-1,0)$ is now not difficult. A simple way is as follows: We choose a continuous extension operator from $W^{1,p}(\omega\times(-1,0))$ to $W^{1,p}(\omega\times(-1,+\infty))$ and define, for all n, a function w_n as the extension of $(u_n|_{\omega\times(-1,0)} - u^0)$. Clearly, $w_n \to 0$ strongly in $W^{1,p}(\omega\times(-1,+\infty))$. The sequence u_n is then modified in the following way: We replace u_n with $u_n - w_n$ in Ω_{f_n}, letting it keep the value 0 outside. This new u_n satisfies the same properties as before, but additionally, $u_n = u^0$ a.e. in $\omega\times(-1,0)$. This shows the thesis. ☐

5. An approximation result. We introduce in this section, as in [9], a phase-field approximation of the functional $\overline{F}$. The idea is to represent the subgraph $\Omega_h\setminus\Sigma$ by a field v that will be an approximation of the characteristic function of this set, at a scale of order ε. Then, numerically, the minimization of our new functional will provide an approximation of (u,h) minimizing $\overline{F}$. Our approximated functional is the following:

$$F_\varepsilon(u,v) = \int_{\omega\times(0,+\infty)} (\eta_\varepsilon + v^2(x))W(\nabla u(x))\,dx + c_V\left(\frac{\varepsilon}{2}\int_{\omega\times(0,+\infty)} |\nabla v(x)|^2\,dx + \frac{1}{\varepsilon}\int_{\omega\times(0,+\infty)} V(v(x))\,dx\right) \tag{29}$$

if $u \in W^{1,p}(\omega\times(0,+\infty))$, with $u=u^0$ on $\omega\times\{0\}$, and $v\in H^1(\omega\times(0,+\infty))$, with $v=1$ on $\omega\times\{0\}$ and $\partial_N v \leq 0$ a.e. in $\omega\times(0,+\infty)$. Otherwise, for all other $u,v\in L^1(\omega\times(0,+\infty))$, we let $F_\varepsilon(u,v)=+\infty$. Here the potential V is a two-wells potential with $V(t) > 0$ except if $t \in \{0,1\}$, $V(0) = V(1) = 0$, and $c_V^{-1} = \int_0^1 \sqrt{2V(t)}\,dt$. The parameter η_ε is any function of ε with $\eta_\varepsilon/(\varepsilon^{p-1}) \to 0$ as $\varepsilon \to 0$. The function u^0 is assumed to be the trace of a function in $W^{1,p}(\omega\times(-1,0))$, still denoted by u^0, and for technical reasons we also have to assume that it is bounded: $u^0 \in L^\infty(\omega\times(-1,0))$. The following results generalize in arbitrary dimension Theorem 3.1 of [9]. However, its proof also owes a lot to [13, sect. 5.2], where a similar approximation is studied.

THEOREM 5.1. *Let $(\varepsilon_j)_{j\ge 1}$ be a decreasing sequence of positive numbers, going to 0. Then the following hold.*

(i) *For any (u_j, v_j), if $\limsup_{j\to\infty} F_{\varepsilon_j}(u_j,v_j) < +\infty$, then up to a subsequence there exist u, v such that $v_j \to v$ in $L^1(\omega\times(0,+\infty))$ and $u_j(x) \to u(x)$ a.e. in $\{v = 1\}$, and there exists $h \in BV(\omega;\mathbb{R}_+)$ such that $\{v=1\} = \Omega_h$ and*

$$\overline{F}(u,h) \;\le\; \liminf_{j\to\infty} F_{\varepsilon_j}(u_j,v_j)\,. \tag{30}$$

(ii) *For any $h \in BV(\omega;\mathbb{R}_+)$ and $u \in GSBV_p(\omega\times(-1,+\infty))$ with $u = u^0$ in $\omega\times(-1,0)$ and $u(x) = 0$ a.e. in $\{x_N > h(x')\}$, there exists (u_j,v_j) such that $u_j \to u$ and $v_j \to \chi_{\Omega_h}$ in $L^1(\omega\times(0,+\infty))$ and*

$$\limsup_{j\to\infty} F_{\varepsilon_j}(u_j,v_j) \;\le\; \overline{F}(u,h)\,. \tag{31}$$

This is almost a Γ-convergence result. We deduce, in particular, that if for all j, (u_j,v_j) is a minimizer of F_{ε_j}, then up to a subsequence, $v_j \to \chi_{\Omega_h}$ and $u_j \to u$ a.e. in Ω_h, where (u,h) minimize the relaxed functional $\overline{F}$.

Remark 5.2. The thesis of the theorem is still valid if (as in [9, Thm. 3.1]) the set Ω_h must satisfy a volume constraint $|\Omega_h| = V > 0$ (which is imposed in the approximation by a constraint on v_j: $\int_{\omega\times(0,+\infty)} v_j(x)\,dx = V$). The adaption of the proofs is easy; see Remark 4.2 above.

Proof of Theorem 5.1. We first show the first point. Let (u_j,v_j) be as in (i). Since $F_{\varepsilon_j}(u_j,v_j)$ is finite, v_j must be nondecreasing in x_N. Now, if we replace v_j by $\tilde v_j(x) = 0 \vee ((v_j(x) - \delta_j x_N)\wedge 1)$ and if δ_j is small enough, one can ensure that $F_{\varepsilon_j}(u_j,v_j) = F_{\varepsilon_j}(u_j,\tilde v_j) + O(1/j)$, and $\tilde v_j$ is strictly decreasing.

Assume first that v_j is smooth, so that $\tilde v_j$ is smooth in $\{0 < \tilde v_j < 1\}$. For any $s \in (0,1)$, let $h_j^s : \omega \to \mathbb{R}_+$ be the function such that $\tilde v_j(x', h_j^s(x')) = s$ for any $x' \in \omega$; then clearly, h_j^s is in $C^1(\omega)$, with

$$|\nabla' h_j^s(x')| \;=\; \frac{|\nabla'\tilde v_j(x', h_j^s(x'))|}{|\partial_N \tilde v_j(x', h_j^s(x'))|} \;\le\; \frac{1}{\delta_j}|\nabla'\tilde v_j(x', h_j^s(x'))|$$

for any $x' \in \omega$. Now, we deduce that

$$\begin{aligned}\int_\omega |\nabla' h_j^s(x')|^2\,dx' \;&\le\; \frac{1}{\delta_j}\int_\omega \frac{|\nabla'\tilde v_j(x', h_j^s(x'))|^2}{|\partial_N \tilde v_j(x', h_j^s(x'))|}\,dx' \\ &\le\; \frac{1}{\delta_j}\int_\omega |\nabla'\tilde v_j(x', h_j^s(x'))|^2 \left(\frac{\sqrt{1+|\nabla' h_j^s(x')|^2}}{|\nabla\tilde v_j(x', h_j^s(x'))|}\right)dx' \\ &=\; \frac{1}{\delta_j}\int_{\partial\{\tilde v_j > s\}} \frac{|\nabla'\tilde v_j(x)|^2}{|\nabla\tilde v_j(x)|}\,d\mathcal{H}^{N-1}(x)\,.\end{aligned}$$

Using the coarea formula, we find that

$$\int_0^1 \left(\int_\omega |\nabla' h_j^s(x')|^2 \, dx' \right) ds \;\leq\; \frac{1}{\delta_j} \int_{\{1>\tilde{v}_j>0\}} |\nabla' \tilde{v}_j(x)|^2 \, dx \;<\; +\infty\,.$$

By approximation, we easily deduce that this remains true when v_j is just in $H^1(\omega \times (0,+\infty))$: We get that for a.e. level $s \in (0,1)$, the set $\{\tilde{v}_j > s\}$ can be represented as the subgraph of a function $h_j^s \in H^1(\omega)$. We may also assume that this is true for all $j \geq 1$.

Now, we notice that (using $a^2 + b^2 \geq 2ab$ and the coarea formula)

$$(32)\quad \frac{\varepsilon_j}{2} \int_{\omega\times(0,+\infty)} |\nabla \tilde{v}_j(x)|^2 \, dx + \frac{1}{\varepsilon_j} \int_{\omega\times(0,+\infty)} V(\tilde{v}_j(x)) \, dx$$
$$\geq \int_{\omega\times(0,+\infty)} \sqrt{2V(\tilde{v}_j(x))} |\nabla \tilde{v}_j(x)| \, dx$$
$$\geq \int_0^1 \sqrt{2V(s)} \left(\int_\omega \sqrt{1 + |\nabla' h_j^s(x')|^2} \, dx' \right)$$

and, in particular, using Fatou's lemma, we see that

$$\int_0^1 \sqrt{2V(s)} \left(\liminf_{j\to\infty} \int_\omega \sqrt{1 + |\nabla' h_j^s(x')|^2} \, dx' \right)$$
$$\leq \liminf_{j\to\infty} \left(\frac{\varepsilon_j}{2} \int_{\omega\times(0,+\infty)} |\nabla \tilde{v}_j(x)|^2 \, dx + \frac{1}{\varepsilon_j} \int_{\omega\times(0,+\infty)} V(\tilde{v}_j(x)) \, dx \right).$$

In particular, for a.e. $s \in (0,1)$, $h_j^s \in H^1(\omega)$ for all $j \geq 1$, and in addition, $\liminf_{j\to\infty} \sqrt{1 + |\nabla' h_j^s|^2}$ is finite.

By a diagonal argument, we can find a subsequence (still denoted by (ε_j)) and a decreasing sequence $(s_n)_{n\geq 1}$ of real numbers in $(0,1)$ with $\lim_{n\to\infty} s_n = 0$ and such that, for each n,

$$\lim_{j\to\infty} \int_\omega \sqrt{1 + |\nabla' h_j^{s_n}(x')|^2} \, dx' \;=\; \liminf_{j\to\infty} \int_\omega \sqrt{1 + |\nabla' h_j^{s_n}(x')|^2} \, dx' \;<\; +\infty\,.$$

We can also assume that, for each n, $h_j^{s_n}$ converges in $L^1(\omega)$ to some function h^{s_n}, and since it is then clear (since $V(\tilde{v}_j(x)) \to 0$ a.e. in $\omega \times (0,+\infty)$) that $\tilde{v}_j(x) \to 0$ for a.e. x with $x_N > h^{s_n}(x')$ and $\tilde{v}_j(x) \to 1$ for a.e. x with $x_N < h^{s_n}(x')$, this function is independent on n and will be denoted simply by h.

For any $n \geq 1$, let us denote by u_j^n the function given by $u_j(x)$ if $x_N < h_j^{s_n}(x')$ and by 0 otherwise; let us show that $(u_j^n)_{j\geq 1}$ is compact in $GSBV(\omega \times (-1,+\infty))$. One has $u_j^n \in W^{1,p}(\{x \,:\, -1 < x_N < h_j^{s_n}(x')\})$; hence, $u_j^n \in GSBV(\omega \times (-1,+\infty))$ with $S_{u_j^n} \subseteq \{(x', h_j^{s_n}(x')) \,:\, x' \in \omega\}$. In particular,

$$\mathcal{H}^{N-1}(S_{u_j^n}) \;\leq\; \int_\omega \sqrt{1 + |\nabla' h_j^{s_n}(x')|^2} \, dx'$$

is uniformly bounded (in j). On the other hand,

$$F_{\varepsilon_j}(u_j, \tilde{v}_j) \;\geq\; (\eta_{\varepsilon_j} + s_n^2) \int_{\omega\times(0,+\infty)} W(\nabla u_j^n(x)) \, dx$$

showing that ∇u_j^n is uniformly bounded in $L^p(\omega \times (-1,+\infty); \mathbb{R}^N)$.

Now, for any $x' \in \omega$, if we denote by $\hat{u}_j^n$ the function $u_j^n - u^0$ (where u^0 is appropriately extended to a function in $W^{1,p}(\omega \times (-1,+\infty))$ that vanishes for $x_N \geq 1$), one sees that, for any x with $x_N < h_j^{s_n}(x')$,

$$|\hat{u}_j^n(x)| \;\leq\; \int_0^{x_N} |\partial_N \hat{u}_j^n(x',s)|\,ds \;\leq\; x_N^{1-1/p}\left(\int_0^{x_N} |\partial_N \hat{u}_j^n(x',s)|^p\,ds\right)^{1/p}$$

so that, for any $M > 0$ and a.e. $x' \in \omega$,

$$\int_0^{M\wedge h_j^{s_n}(x')} |\hat{u}_j^n(x',s)|\,ds \;\leq\; \frac{M^{2-1/p}}{2^{1-1/p}}\left(\int_0^{h_j^{s_n}(x')} |\partial_N \hat{u}_j^n(x',s)|^p\,ds\right)^{1/p}.$$

We get

$$\|\hat{u}_j^n\|_{L^1(\omega\times(-1,M))} \;\leq\; C(M)\|\partial_N \hat{u}_j^n\|_{L^p(\omega\times(-1,+\infty))}.$$

Therefore, $u_j^n = \hat{u}_j^n + u^0$ is uniformly bounded in $L^1_{\text{loc}}(\omega \times (-1,+\infty))$. By Ambrosio's compactness theorem we deduce that there exists $u^n \in GSBV_p(\omega \times (-1,+\infty))$ such that $u_j^n(x) \to u^n(x)$ a.e. in $\omega \times (-1,+\infty)$, up to a subsequence.

By a diagonal argument, we can extract a subsequence (still denoted by $(\varepsilon_j)_{j\geq 1}$) such that for each $n \geq 0$, $u_j^n(x) \to u^n(x)$ a.e. as $\varepsilon_j \to 0$. Now, by construction we have that if $n' \geq n$, then $u_j^{n'}(x) = u_j^n(x)$ a.e. in $\{x_N < h_j^n(x')\}$. From this we deduce that $u^{n'}(x) = u^n(x)$ a.e. in $\{x_N < h(x')\}$, and since moreover one checks easily that both functions vanish a.e. in $\{x_N > h(x')\}$, one deduces that u^n, which is simply denoted by u in the following, is independent on n.

We have shown the first assertion of point (i) of Theorem 5.1: Indeed, if we let $v = \chi_{\Omega_h}$, one sees that $\tilde{v}_j(x) \to v(x)$ a.e., and by construction also $v_j(x) \to v(x)$ a.e. in $\omega \times (0,+\infty)$. Moreover, $u_j(x) \to u(x)$ a.e. in $\{x \in \omega \times (-1,+\infty) : x_N < h(x)\}$, with $u = u^0$ in $\omega \times (-1,0)$. The function u is in $GSBV_p(\omega \times (-1,+\infty))$ and vanishes above the graph of h.

Let us now show (30). We follow a similar proof as that in [13]. We have

$$\begin{aligned}\int_{\omega\times(0,+\infty)} (\eta_{\varepsilon_j} + \tilde{v}_j^2(x))W(\nabla u_j(x))\,dx \;&\geq\; \int_{\omega\times(0,+\infty)}\left(2\int_0^{\tilde{v}_j(x)} s\,ds\right)W(\nabla u_j(x))\,dx\\ &\geq\; \int_0^1 2s\left(\int_{\{\tilde{v}_j(x)>s\}} W(\nabla u_j(x))\,dx\right)ds.\end{aligned}$$

This inequality, together with (32), yields

$$\begin{aligned}&F_{\varepsilon_j}(u_j,\tilde{v}_j) \;\geq\\ &\quad\int_0^1 \left(2s\int_{\{\tilde{v}_j(x)>s\}} W(\nabla u_j(x))\,dx \;+\; c_V\sqrt{2V(s)}\int_\omega \sqrt{1+|\nabla' h_j^s(x')|^2}\,dx'\right)ds.\end{aligned}$$

By Fatou's lemma, we deduce that

(33)
$$\begin{aligned}&\int_0^1 \liminf_{j\to\infty}\left(2s\int_{\{\tilde{v}_j(x)>s\}} W(\nabla u_j(x))\,dx \;+\; c_V\sqrt{2V(s)}\int_\omega \sqrt{1+|\nabla' h_j^s(x')|^2}\,dx'\right)ds\\ &\qquad\qquad\leq\; \liminf_{j\to\infty} F_{\varepsilon_j}(u_j,\tilde{v}_j) \;<\; +\infty.\end{aligned}$$

Therefore, for a.e. $s \in (0,1)$,

$$\liminf_{j\to\infty}\ 2s\int_{\{\tilde v_j(x)>s\}} W(\nabla u_j(x))\,dx \,+\, c_V\sqrt{2V(s)}\int_\omega \sqrt{1+|\nabla' h_j^s(x')|^2}\,dx' \ <\ +\infty\,.$$

Let us choose such an s, with additionally $h_j^s \in H^1(\omega)$ for all $j\ge 1$, and let us consider a subsequence $(j_k)_{k\ge1}$ such that

$$\lim_{k\to\infty} 2s\int_{\{\tilde v_{j_k}(x)>s\}} W(\nabla u_{j_k}(x))\,dx \,+\, c_V\sqrt{2V(s)}\int_\omega \sqrt{1+|\nabla' h_{j_k}^s(x')|^2}\,dx'$$
$$=\ \liminf_{j\to\infty}\ 2s\int_{\{\tilde v_j(x)>s\}} W(\nabla u_j(x))\,dx \,+\, c_V\sqrt{2V(s)}\int_\omega \sqrt{1+|\nabla' h_j^s(x')|^2}\,dx'\,.$$

As above, let us introduce the sequence of functions $u^s_{j_k} \in GSBV_p(\omega\times(-1,+\infty))$ such that $u^s_{j_k}(x) = u_{j_k}(x)$ if $x_N < h^s_{j_k}(x')$ and 0 otherwise. By compactness, we easily check that $u^s_{j_k}(x) \to u(x)$ a.e. in $\omega\times(-1,+\infty)$, while $h^s_{j_k} \to h$ in $L^1(\omega)$. By the l.s.c. property (P1), we deduce

$$2s\int_{\Omega_h^+} W(\nabla u) \,+\, c_V\sqrt{2V(s)}(\mathcal{H}^{N-1}(\partial_*\Omega_h) + 2\mathcal{H}^{N-1}(S'_u\cap\Omega_h^1))$$
$$\le\ \lim_{k\to\infty} 2s\int_{\{\tilde v_{j_k}(x)>s\}} W(\nabla u_{j_k}(x))\,dx \,+\, c_V\sqrt{2V(s)}\int_\omega \sqrt{1+|\nabla' h_{j_k}^s(x')|^2}\,dx'\,.$$

Integrating (33) on $(0,1)$ and recalling that by construction $F_{\varepsilon_j}(u_j,\tilde v_j) = F_{\varepsilon_j}(u_j,v_j) + o(1)$, we deduce (30).

Let us now show point (ii) of Theorem 5.1. The proof follows the same lines as in [9], where the same inequality is shown in the 2D case, and we will only sketch it.

Let $h\in BV(\omega;\mathbb{R}_+)$, and let $u\in GSBV_p(\omega\times(-1,+\infty))$, with $u=u^0$ in $\omega\times(-1,0)$ and $u(x)=0$ a.e. in $\{x_N>h(x')\}$, with $\overline{F}(u,h)<+\infty$. By Theorem 2.2, there exist h_n in $C^1(\omega;\mathbb{R}_+)$ and $u^n\in W^{1,p}(\Omega_h;\mathbb{R})$, with $u^n=u^0$ in $\omega\times(-1,0)$, $h_n\to h$ in $L^1(\omega)$, and $u^n\to u$ a.e. in $\omega\times(0,+\infty)$, with

$$\limsup_{n\to\infty} F(u^n,h_n) = \overline{F}(u,h)\,.$$

By construction (since we have assumed $u^0\in L^\infty(\omega\times(-1,0))$), one also has that $u^n\in L^\infty(\omega\times(0,+\infty))$. Now, we construct sequences $(u^n_j)_j$ and $(v^n_j)_j$ with $u^n_j\to u^n$ in $L^1(\omega\times(0,+\infty))$ and $v^n_j\to\chi_{\Omega_{h_n}}$ in $L^1(\omega\times(0,+\infty))$ such that

$$\limsup_{j\to\infty} F_{\varepsilon_j}(u^n_j,v^n_j)\le F(u^n,h_n). \tag{34}$$

Let us condider the sequence of functionals

$$H_\varepsilon(v) = \frac{\varepsilon}{2}\int_{\omega\times(0,+\infty)} |\nabla v(x)|^2\,dx + \frac{1}{\varepsilon}\int_{\omega\times(0,+\infty)} V(v(x))\,dx;$$

the celebrated Γ-convergence result of Modica and Mortola for such functionals (see [1]) allows us to find, for each n, a sequence $(v^n_j)_j$ converging to the characteristic function $\chi_{\Omega_{h_n}}$ such that

$$\limsup_{j\to\infty} H_{\varepsilon_j}(v^n_j) = \int_0^1 \sqrt{2V(s)}\,ds\ \mathcal{H}^{N-1}(S_{\chi_{\Omega_{h_n}}}\cap\omega\times(0,+\infty))$$
$$= c_V^{-1}\mathcal{H}^{N-1}(\partial\Omega_{h_n}).$$

We recall that the explicit construction of the recovery sequence $(v_j^n)_j$ can be obtained in the following way: One considers γ_j solution of the Euler's equation of the functional with appropriate boundary conditions, namely,

$$\begin{cases} -\gamma_j'' + V'(\gamma_j) = 0, \\ \gamma_j(0) = 1, \quad \gamma_j\Big(\dfrac{1}{\sqrt{\varepsilon_j}}\Big) = 0\,. \end{cases}$$

This function is extended by 0 beyond $1/\sqrt{\varepsilon_j}$. One then lets

$$v_j^n(x) = \gamma_j\Big(\frac{\operatorname{dist}(x, \Omega_{h_n}^+)}{\varepsilon_j}\Big).$$

Then, the sequence $(u_j^n)_j$ is constructed by translating u_n and multiplying by an appropriate cut-off function, as in [9]. We first choose $c_n \geq \max\{1, \|\nabla h_n\|_{L^\infty(\omega)}\}$, and let $w_j^n(x) := v_j^n(x', x_N - c_n\sqrt{2\varepsilon_j})$. This function is 1 on the support of v_j^n and vanishes shortly beyond. Then, we let $u_j^n(x) = u^n(x', x_N - 2c_n\sqrt{2\varepsilon_j})w_j^n(x)$. (As in the end of the proof of Proposition 4.1, we have to modify slightly u_j^n in order to ensure $u_j^n = u^0$ in $\omega \times (-1,0)$; however, this is easily done, and one checks that this modified u_j^n satisfies a uniform (in j) L^∞ bound.) In order to show that (34) holds, we just need to check

$$\limsup_{j\to\infty} \int_{\omega\times(0,+\infty)} (\eta_{\varepsilon_j} + (v_j^n(x))^2) W(\nabla u_j^n(x))\, dx \;\leq\; \int_{\Omega_{h_n}^+} W(\nabla u^n(x))\, dx\,. \tag{35}$$

Since $\nabla u_j^n(x) = w_j^n(x)\nabla u^n(x', x_N - 2c_n\sqrt{2\varepsilon_j}) + u^n(x', x_N - 2c_n\sqrt{2\varepsilon_j})\nabla w_j^n(x)$, this inequality is clear as soon as we have established that

$$\limsup_{j\to\infty}\; \eta_{\varepsilon_j} \int_{\omega\times(0,+\infty)} |u^n(x', x_N - 2c_n\sqrt{2\varepsilon_j})\nabla w_j^n(x)|^p\, dx = 0,$$

and since u^n is bounded in L^∞, we need to show

$$\limsup_{j\to\infty}\; \eta_{\varepsilon_j} \int_{\omega\times(0,+\infty)} |\nabla w_j^n(x)|^p\, dx = 0\,.$$

This integral is bounded by

$$\begin{aligned} &\int_{\{0<\operatorname{dist}(x,\Omega_{h_n}^+)<\sqrt{\varepsilon_j}\}} \frac{|\gamma_j'|^p\left(\operatorname{dist}(x,\Omega_{h_n}^+)/\varepsilon_j\right)}{\varepsilon_j^p}\, dx \\ &\qquad = \int_0^{\sqrt{\varepsilon_j}} \frac{|\gamma_j'|^p(s/\varepsilon_j)}{\varepsilon_j^p} \mathcal{H}^{N-1}(\{\operatorname{dist}(\cdot, \Omega_{h_n}^+) = s\})\, ds \\ &\qquad\qquad = \frac{1}{\varepsilon_j^{p-1}} \int_0^{1/\sqrt{\varepsilon_j}} |\gamma_j'|^p(s) \mathcal{H}^{N-1}(\{\operatorname{dist}(\cdot, \Omega_{h_n}^+) = \varepsilon_j s\})\, ds\,. \end{aligned}$$

Now, one can show that

$$\int_0^{1/\sqrt{\varepsilon_j}} |\gamma_j'|^p(s)\mathcal{H}^{N-1}(\{\operatorname{dist}(\cdot, \Omega_{h_n}^+) = \varepsilon_j s\})\, ds \to \mathcal{H}^{N-1}(\partial\Omega_{h_n}^+) \int_0^1 \sqrt{2V(t)}^{\,p-1}\, dt$$

as $j \to \infty$; hence, since we have assumed $\eta_\varepsilon/\varepsilon^{p-1} \to 0$ as $\varepsilon \to 0$, we deduce (35) and (34).

Since (34) holds, a standard diagonal extraction argument allows us to find subsequences $(u_{j_k}^{n_k})_k$, $(v_{j_k}^{n_k})_k$ satisfying point (ii) of Theorem 5.1, and this completes the proof of the theorem.

Appendix A. A simple inequality.

LEMMA A.1. *Let $w \in C^1([0,1]^N)$ satisfy, for any $x \in [0,1]^N$,*

$$w(x) \;=\; \sum_{k\in\{0,1\}^N} w(k)\Delta(x-k),$$

where Δ is defined in (21)*. Then, for any $p \geq 1$,*

$$\int_{(0,1)^N} W_p(\nabla w(x))\,dx \;\leq\; \frac{1}{2^{N-1}}\sum_{i=1}^{N}\sum_{\substack{k\in\{0,1\}^N\\ k_i=0}} |w(k+e_i)-w(k)|^p. \tag{36}$$

Proof. We show that, for each i,

$$\int_{(0,1)^N}\left|\frac{\partial w}{\partial x_i}(x)\right|^p dx \;\leq\; \frac{1}{2^{N-1}}\sum_{\substack{k\in\{0,1\}^N\\ k_i=0}} |w(k+e_i)-w(k)|^p\,.$$

We will show this inequality for $i = N$. Let us denote, for $x' = (x_1,\dots,x_{N-1})$,

$$\Delta_{N-1}(x') \;=\; \prod_{i=1}^{N-1}(1-|x_i|)^+\,.$$

Then, for any $x \in (0,1)^N$,

$$\begin{aligned} w(x) \;&=\; \sum_{k\in\{0,1\}^N} w(k)\Delta(x-k)\\ &=\; \sum_{k'\in\{0,1\}^{N-1}} \Delta_{N-1}(x'-k')(w_{k',0}(1-x_N)+w_{k',1}x_N) \end{aligned}$$

so that

$$\frac{\partial w}{\partial x_N}(x) \;=\; \sum_{k'\in\{0,1\}^{N-1}} \Delta_{N-1}(x'-k')(w_{k',1}-w_{k',0})\,.$$

Now, at any x, we have $\sum_{k'\in\{0,1\}^{N-1}}\Delta_{N-1}(x'-k') = 1$ so that this is a convex combination of $(w_{k',1}-w_{k',0})_{k'\in\{0,1\}^{N-1}}$. Hence, by convexity of the function $|\cdot|^p$,

$$\int_{(0,1)^N}\left|\frac{\partial w}{\partial x_N}(x)\right|^p dx \;\leq\; \int_{(0,1)^N}\sum_{k'\in\{0,1\}^{N-1}}\Delta_{N-1}(x'-k')|w_{k',1}-w_{k',0}|^p\,dx\,.$$

We deduce (36) by simply observing that, for any $k' \in \{0,1\}^{N-1}$,

$$\int_{(0,1)^N}\Delta_{N-1}(x'-k')\,dx \;=\; \int_0^1 dx_N \times \prod_{i=1}^{N-1}\int_0^1 (1-|x_i-k_i|)^+\,dx_i \;=\; \frac{1}{2^{N-1}}\,. \qquad \square$$

Acknowledgments. The authors would like to thank A. Braides, E. Bonnetier, and F. Jouve for stimulating discussions and the two latter for providing the result of the numerical experiment shown in Figure 1. The authors are also grateful to the anonymous referees, whose comments helped improve a previous version of this work.

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