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# *Long-Time Asymptotics of a Multiscale Model for Polymeric Fluid Flows*

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*Communicated by the Editors*

**Abstract**

In this paper, we investigate the long-time behavior of some micro-macro models for polymeric fluids (Hookean model and FENE model), in various settings (shear flow, general bounded domain with homogeneous Dirichlet boundary conditions on the velocity, general bounded domain with non-homogeneous Dirichlet boundary conditions on the velocity). We use both probabilistic approaches (coupling methods) and analytic approaches (entropy methods).

## 1. Introduction

We are interested in the long-time behavior of some micro-macro models for dilute solutions of polymers. Such models couple a macroscopic description of the flow with a mesoscopic description of the dynamics of the polymer chains. More precisely, the motion of the polymer chains in the fluid is modelled by a kinetic equation (Langevin dynamics) and an averaging procedure enables derivation of the contribution of the polymer chains to the stress tensor within the fluid. The evolution of the velocity field in the fluid is then described by the classical laws of conservation of momentum and mass. The physical and mechanical background can be read in the following textbooks: [5, 6, 11, 27].

Mathematically, the system reads (in a non-dimensional form):

$$\mathrm{Re}\left(\frac{\partial \boldsymbol{u}}{\partial t}(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}).\nabla \boldsymbol{u}(t,\boldsymbol{x})\right) = (1-\varepsilon)\Delta \boldsymbol{u}(t,\boldsymbol{x}) - \nabla p(t,\boldsymbol{x}) + \mathrm{div}\ \boldsymbol{\tau}(t,\boldsymbol{x}), \tag{1}$$

$$\mathrm{div}\ (\boldsymbol{u}(t,\boldsymbol{x})) = 0, \tag{2}$$

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \frac{\varepsilon}{\mathrm{We}}\left(\int_{\mathbb{R}^d} (\boldsymbol{X}\otimes\nabla\Pi(\boldsymbol{X}))\psi(t,\boldsymbol{x},\boldsymbol{X})\,d\boldsymbol{X} - \mathrm{Id}\right), \tag{3}$$

$$
\begin{aligned}
\frac{\partial \psi}{\partial t}(t, \boldsymbol{x}, \boldsymbol{X}) &+ \boldsymbol{u}(t, \boldsymbol{x}).\nabla_{\boldsymbol{x}} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \\
&= -\mathrm{div}_{\boldsymbol{X}} \left( (\nabla_{\boldsymbol{x}} \boldsymbol{u}(t, \boldsymbol{x}) \boldsymbol{X} - \frac{1}{2\mathrm{We}} \nabla \Pi(\boldsymbol{X})) \psi(t, \boldsymbol{x}, \boldsymbol{X}) \right) \\
&\quad + \frac{1}{2\mathrm{We} M} \Delta_{\boldsymbol{X}} \psi(t, \boldsymbol{x}, \boldsymbol{X}),
\end{aligned}
\tag{4}
$$

where the Reynolds number $\mathrm{Re} > 0$, the Weissenberg number $\mathrm{We} > 0$, $\varepsilon \in (0, 1)$ and $M > 0$ are the non-dimensional numbers in the system. The Reynolds number expresses the ratio of inertial forces to viscous forces in the fluid. The Weissenberg number is the ratio of the characteristic time of the polymer chain to the characteristic time of the fluid. The non-dimensional number $\varepsilon$ is the ratio of the viscosity due to the polymer chains to the total viscosity. The non-dimensional number $M$ is the square of the characteristic length of the domain where $\boldsymbol{x}$ varies to the characteristic length of the polymer chains.

We suppose that the space variable $\boldsymbol{x}$ varies in *a regular and bounded domain* $\mathcal{D}$ of $\mathbf{R}^d$, with $d = 2$ or 3. Appropriate initial conditions are inputted into this system together with appropriate boundary conditions on the velocity $\boldsymbol{u}$ and on the distribution $\psi$. At this stage of the discussion, we will not give precise values for any of these variables.

In the considered model, the polymer chain is approximated by two beads linked by their end-to-end vector $\boldsymbol{X}$ (this is the dumbbell model: see Fig. 1). By writing a Langevin equation on each bead, with some approximation (see the Fokker-Planck equation (4) for the distribution $\psi$ of $\boldsymbol{X}$ at time $t$ and at point $\boldsymbol{x}$), it is possible to derive a kinetic equation for $\boldsymbol{X}$ which includes the velocity field $\boldsymbol{u}$. In return, the polymer chain influences the flow field through an extra stress tensor $\boldsymbol{\tau}$, which appears in the right-hand side of the momentum equation (1).

The vector $\nabla \Pi(\boldsymbol{X})$ in (4) is the force (of entropic origin) between the two beads. In this paper, we will specifically consider two forms of potential $\Pi$:

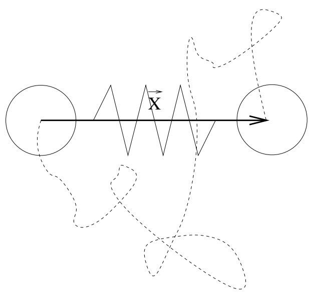


**Fig. 1.** In the dumbbell model, the polymer (represented as a dashed line) is modelled as two beads linked by a spring. The length and the orientation of the polymer are given by the so-called end-to-end vector $\boldsymbol{X}$.

(i) Hookean dumbbells:

$$\Pi(\boldsymbol{X}) = \frac{|\boldsymbol{X}|^2}{2}, \tag{5}$$

(ii) Finite Extensible Nonlinear Elastic (FENE) dumbbells:

$$\Pi(\boldsymbol{X}) = -\frac{bM}{2} \ln\left(1 - \frac{|\boldsymbol{X}|^2}{bM}\right), \tag{6}$$

where $b$ is a non-dimensional parameter related to the maximal length of the polymer chain. The FENE model (6) takes into account the finite extensibility of the polymer chain, through an explosive force when $|\boldsymbol{X}|$ tends to $\sqrt{bM}$.

**Remark 1** (The $\alpha$-convexity of $\Pi$**).** Many of the results we present (for example those in Section 3.2.2) are actually true for any potential which is a $\alpha$-convex function (with $\alpha > 0$), i.e., a function satisfying, for all $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathbf{R}^d$, and for all $\lambda \in (0, 1)$,

$$\Pi(\lambda \boldsymbol{X} + (1-\lambda)\boldsymbol{Y}) \leqq \lambda \Pi(\boldsymbol{X}) + (1-\lambda)\Pi(\boldsymbol{Y}) - \frac{\alpha\lambda(1-\lambda)}{2}|\boldsymbol{X} - \boldsymbol{Y}|^2, \tag{7}$$

and which can be expressed as a function of the norm of $\boldsymbol{X}$ (i.e., a radially-symmetric function):

$$\Pi(\boldsymbol{X}) = \pi(|\boldsymbol{X}|). \tag{8}$$

The $\alpha$-convexity of $\Pi$ can be shown to be equivalent to the $\alpha$-convexity of $\pi$ together with $\pi'(0) \geqq 0$. These properties on $\pi$ can be easily checked for the case of Hookean and FENE dumbbells (with $\alpha = 1$ for both of these models). These two features of $\Pi$ (radial symmetry and $\alpha$-convexity) are key properties, used many times in this paper. Here we concentrate on the two potentials (5) and (6), since they are prototypical of those used in the rheological literature.

The purpose of the present paper is to study the long-time limit of the fields $(\boldsymbol{u}, \psi, \boldsymbol{\tau})$. It is of course expected that, under appropriate conditions, they converge, (see later for a detailed discussion) to some fields $(\boldsymbol{u}_\infty, \psi_\infty, \boldsymbol{\tau}_\infty)$ which $\Delta$ satisfy:

$$\mathrm{Re}\boldsymbol{u}_\infty(\boldsymbol{x}).\nabla \boldsymbol{u}_\infty(\boldsymbol{x}) = (1-\varepsilon)\Delta \boldsymbol{u}_\infty(\boldsymbol{x}) - \nabla p_\infty(\boldsymbol{x}) + \mathrm{div}\, \boldsymbol{\tau}_\infty(\boldsymbol{x}), \tag{9}$$

$$\mathrm{div}\,(\boldsymbol{u}_\infty) = 0, \tag{10}$$

$$\boldsymbol{\tau}_\infty(\boldsymbol{x}) = \frac{\varepsilon}{\mathrm{We}}\left(\int_{\mathbf{R}^d} (\boldsymbol{X} \otimes \nabla\Pi(\boldsymbol{X}))\psi_\infty(\boldsymbol{x}, \boldsymbol{X})\, d\boldsymbol{X} - \mathrm{Id}\right), \tag{11}$$

$$\boldsymbol{u}_\infty(\boldsymbol{x}).\nabla_{\boldsymbol{x}}\psi_\infty(\boldsymbol{x}, \boldsymbol{X}) = -\mathrm{div}_{\,\boldsymbol{X}}\left((\nabla_{\boldsymbol{x}}\boldsymbol{u}_\infty(\boldsymbol{x})\boldsymbol{X} - \frac{1}{2\mathrm{We}}\nabla\Pi(\boldsymbol{X}))\psi_\infty(\boldsymbol{x}, \boldsymbol{X})\right) + \frac{1}{2\mathrm{We}M}\Delta_{\boldsymbol{X}}\psi_\infty(\boldsymbol{x}, \boldsymbol{X}). \tag{12}$$

In order to prove this convergence mathematically, we will mainly use the so-called entropy method. Considering, as an example, the Fokker-Planck equation (4) for a null velocity field $\boldsymbol{u} = 0$, the idea is to introduce the relative entropy

$H(t) = \int_{\mathbf{R}^d} h\left(\frac{\psi}{\psi_\infty}\right)\psi_\infty$ of $\psi$ with respect to $\psi_\infty$, where $h$ is, for example, $h(x) = x\ln(x) - (x-1)$ and $\psi_\infty$ is defined as a stationary solution to (4). By differentiating $H$ with respect to time and using a so-called logarithmic Sobolev inequality (see (39) below) for $\psi_\infty$, an inequality $\frac{dH}{dt} \leqq -CH$ (with $C > 0$) can be proven, which ensures the exponential decay of $H$ to 0. See Section 2.1 for further details and Sections 3.2 and 3.3 for the adaptation of this method to the coupled system (1)–(2)–(3)– (4). The unusual feature when dealing with the coupled system, is that $\psi_\infty$ may not satisfy the detailed balance (see (42) below) nor be explicitly known, contrary to many cases considered in the literature. However, this does not prevent us from completing the proof, at least in some cases.

As an alternative to the entropy method, we will use, whenever possible, a coupling method. To use this method, we introduce the stochastic process $\boldsymbol{X}_t$ which is a solution to the stochastic differential equation associated with the Fokker-Planck equation (4):

$$\begin{aligned} &d\boldsymbol{X}_t(\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}).\nabla_{\boldsymbol{x}}\boldsymbol{X}_t(\boldsymbol{x})\,dt \\ &\quad = \left(\nabla_{\boldsymbol{x}}\boldsymbol{u}(t,\boldsymbol{x})\boldsymbol{X}_t(\boldsymbol{x}) - \frac{1}{2\mathrm{We}}\nabla\Pi(\boldsymbol{X}_t(\boldsymbol{x}))\right)dt + \frac{1}{\sqrt{\mathrm{We}M}}d\boldsymbol{W}_t, \end{aligned} \tag{13}$$

where the law of $\boldsymbol{X}_0(\boldsymbol{x})$ is $\psi(0,\boldsymbol{x},\boldsymbol{X})\,d\boldsymbol{X}$. By considering a stationary stochastic process $\boldsymbol{X}_t^\infty$ with law $\psi_\infty(\boldsymbol{X})\,d\boldsymbol{X}$ which is coupled with $\boldsymbol{X}_t$ (through the driving Brownian motion), it is possible to show the convergence of $(\psi - \psi_\infty)$ to 0 in some appropriate norm. See Section 2.3 for further details and Section 3.1 for an adaptation of this method in the coupled framework.

The results presented here are far from being complete. We hope to stimulate further research in the same direction.

### *1.1. General setting*

Henceforth, for simplification, we take the following values for the non-dimensional parameters: $\mathrm{Re} = \frac{1}{2}$, $\mathrm{We} = 1$, $\varepsilon = \frac{1}{2}$ and $M = 1$. We are thus interested in the following system[1]:

$$\frac{\partial \boldsymbol{u}}{\partial t}(t,\boldsymbol{x}) + \boldsymbol{u}(t,\boldsymbol{x}).\nabla\boldsymbol{u}(t,\boldsymbol{x}) = \Delta\boldsymbol{u}(t,\boldsymbol{x}) - \nabla p(t,\boldsymbol{x}) + \mathrm{div}\,\boldsymbol{\tau}(t,\boldsymbol{x}), \tag{14}$$

$$\mathrm{div}\,(\boldsymbol{u}(t,\boldsymbol{x})) = 0, \tag{15}$$

$$\boldsymbol{\tau}(t,\boldsymbol{x}) = \int_{\mathbf{R}^d}(\boldsymbol{X}\otimes\nabla\Pi(\boldsymbol{X}))\psi(t,\boldsymbol{x},\boldsymbol{X})\,d\boldsymbol{X}, \tag{16}$$

[1] Note that, again for simplification, the expressions of the pressure $p$ and of the stress $\boldsymbol{\tau}$ have been changed going from the initial non-dimensionalized system (1)–(2)–(3)–(4) to (14)–(15)–(16)–(17).

$$\begin{aligned}\frac{\partial \psi}{\partial t}(t, \boldsymbol{x}, \boldsymbol{X}) &+ \boldsymbol{u}(t, \boldsymbol{x}).\nabla_{\boldsymbol{x}} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \\ &= -\operatorname{div}_{\boldsymbol{X}}\left((\nabla_{\boldsymbol{x}} \boldsymbol{u}(t, \boldsymbol{x}) \boldsymbol{X} - \frac{1}{2}\nabla \Pi(\boldsymbol{X}))\psi(t, \boldsymbol{x}, \boldsymbol{X})\right) \\ &\quad + \frac{1}{2}\Delta_{\boldsymbol{X}} \psi(t, \boldsymbol{x}, \boldsymbol{X}). \end{aligned} \tag{17}$$

Note that alternatively we can consider a coupled PDE-SDE system, replacing (16)–(17) by:

$$\boldsymbol{\tau}(t, \boldsymbol{x}) = \mathbf{E}\left(\boldsymbol{X}_t(\boldsymbol{x}) \otimes \nabla \Pi(\boldsymbol{X}_t(\boldsymbol{x}))\right), \tag{18}$$

$$\begin{aligned} d\boldsymbol{X}_t(\boldsymbol{x}) &+ \boldsymbol{u}(t, \boldsymbol{x}).\nabla_{\boldsymbol{x}} \boldsymbol{X}_t(\boldsymbol{x})\, dt \\ &= \left(\nabla_{\boldsymbol{x}} \boldsymbol{u}(t, \boldsymbol{x}) \boldsymbol{X}_t(\boldsymbol{x}) - \frac{1}{2}\nabla \Pi(\boldsymbol{X}_t(\boldsymbol{x}))\right) dt + d\boldsymbol{W}_t, \end{aligned} \tag{19}$$

where $\boldsymbol{W}_t$ denotes a $d$-dimensional standard Brownian motion independent from the initial condition $(\boldsymbol{X}_0(\boldsymbol{x}))_{\boldsymbol{x} \in \mathcal{D}}$ which is such that, $\forall \boldsymbol{x} \in \mathcal{D}$, the law of $\boldsymbol{X}_0(\boldsymbol{x})$ is $\psi(0, \boldsymbol{x}, \boldsymbol{X})\, d\boldsymbol{X}$. In the following, we will mostly consider the coupled system of PDEs (14)–(15)–(16)–(17), but also the system coupling the PDE with the SDE (14)–(15)–(18)–(19).

We now define, with more precision, the boundary conditions on $\psi$, and some requirements on the initial condition $\psi(0, \boldsymbol{x}, .)$. Since $\psi(0, \boldsymbol{x}, .)$ is a density, we require that it is nonnegative and that its integral with respect to $\boldsymbol{X} \in \mathbb{R}^d$ is equal to 1. Then, at least formally (but this may indeed be shown mathematically), we have for any $t \geqq 0$ and for any $\boldsymbol{x} \in \mathcal{D}$, $\psi(t, \boldsymbol{x}, .) \geqq 0$ and $\int_{\mathbb{R}^d} \psi(t, \boldsymbol{x}, \boldsymbol{X})\, d\boldsymbol{X} = 1$. Note that by symmetry in the $\boldsymbol{X}$ variable of (17), it is clear that for all $(t, \boldsymbol{x}, \boldsymbol{X})$, $\psi(t, \boldsymbol{x}, \boldsymbol{X}) = \psi(t, \boldsymbol{x}, -\boldsymbol{X})$, if $\psi(0, \boldsymbol{x}, \boldsymbol{X}) = \psi(0, \boldsymbol{x}, -\boldsymbol{X})$. Since it is a natural physical assumption, we suppose it is satisfied henceforth.

In the case of Hookean dumbbells, it is natural to choose an initial condition $\psi(0, \boldsymbol{x}, .)$ which is Gaussian ($\forall \boldsymbol{x} \in \mathcal{D}$) since $\boldsymbol{X}_t$ is an Ornstein-Uhlenbeck process (by the characteristic method (see (33) and the beginning of Section 2), and as a consequence that the drift term is linear). In this case, $\psi(t, \boldsymbol{x}, .)$ is also Gaussian ($\forall t \geqq 0$ and $\forall \boldsymbol{x} \in \mathcal{D}$). Notice that if $\psi(0, \boldsymbol{x}, .)$ is not Gaussian, then the results we prove for the long-time convergence of $\boldsymbol{u}$ and $\boldsymbol{\tau}$ still hold, so long as uniqueness holds for the time-dependent problem, since $\int_{\mathbb{R}^d} \boldsymbol{X} \otimes \boldsymbol{X} \psi(t, \boldsymbol{x}, \boldsymbol{X})$ only depends on $\int_{\mathbb{R}^d} \boldsymbol{X} \otimes \boldsymbol{X} \psi(0, \boldsymbol{x}, \boldsymbol{X})$ and $\boldsymbol{u}$. Therefore, we can replace the non-Gaussian initial condition by a Gaussian random variable with the same covariance matrix, without changing the values of the macroscopic quantities $(\boldsymbol{u}, p, \boldsymbol{\tau})$. In the case of Hookean dumbbells, we assume that the initial condition is Gaussian and we complement Equation (17) on $\psi$ with a decay condition when $|\boldsymbol{X}| \to \infty$.

In the case of FENE dumbbells, provided that $b \geqq 2$, we know that the stochastic process $\boldsymbol{X}_t$ (that is a solution to (19)) does not cross the boundary in finite time (see [18]), so that $\boldsymbol{X}_t$ is also the process killed or reflected at the boundary of $\mathcal{B}(0, \sqrt{b})$. On the other hand, if $b < 2$, we know that the SDE (19) is ill posed, since it yields many solutions (see [18]). Thus, we suppose in the rest of the paper that $b \geqq 2$, and we combine (17) with a no-flux boundary condition:

$$\left(\left(-\nabla_{\boldsymbol{x}}\boldsymbol{u}(t,\boldsymbol{x})\boldsymbol{X}+\frac{1}{2}\nabla\Pi(\boldsymbol{X})\right)\psi(t,\boldsymbol{x},\boldsymbol{X})+\frac{1}{2}\nabla\psi(t,\boldsymbol{x},\boldsymbol{X})\right).\boldsymbol{n}(\boldsymbol{X})=0$$

(see Appendix B for a more rigorous statement of this no-flux boundary condition). Following the considerations above, we also know that $\psi$ is zero on the boundary $\partial\mathcal{B}(0,\sqrt{b})$. With slight disregard to notation, we denote by $\psi$ the density defined on $\mathcal{B}(0,\sqrt{b})$ and also its extension to $\mathbf{R}^d$ by zero outside of $\mathcal{B}(0,\sqrt{b})$. Notice that, for technical reasons, we will also assume that $\psi_0$ decays as $\exp(-\Pi)$ on the boundary of $\mathcal{B}(0,\sqrt{b})$ (see (B.128)).

The boundary conditions on the velocity $\boldsymbol{u}$ will be precisely defined below. We will consider both homogeneous and nonhomogeneous Dirichlet boundary conditions.

As we are interested in the long-time behaviour of the velocity $\boldsymbol{u}$ and the stress $\boldsymbol{\tau}$ (which are the physical quantities of interest), we introduce the stationary system associated with (14)–(15)–(16)–(17):

$$\boldsymbol{u}_\infty(\boldsymbol{x}).\nabla\boldsymbol{u}_\infty(\boldsymbol{x})=\Delta\boldsymbol{u}_\infty(\boldsymbol{x})-\nabla p_\infty(\boldsymbol{x})+\text{div }\boldsymbol{\tau}_\infty(\boldsymbol{x}), \tag{20}$$

$$\text{div }(\boldsymbol{u}_\infty)=0, \tag{21}$$

$$\boldsymbol{\tau}_\infty(\boldsymbol{x})=\int_{\mathbf{R}^d}(\boldsymbol{X}\otimes\nabla\Pi(\boldsymbol{X}))\psi_\infty(\boldsymbol{x},\boldsymbol{X})\,d\boldsymbol{X}, \tag{22}$$

$$\begin{aligned}\boldsymbol{u}_\infty(\boldsymbol{x}).\nabla_{\boldsymbol{x}}\psi_\infty(\boldsymbol{x},\boldsymbol{X})=&-\text{div}_{\boldsymbol{X}}\left((\nabla_{\boldsymbol{x}}\boldsymbol{u}_\infty(\boldsymbol{x})\boldsymbol{X}-\frac{1}{2}\nabla\Pi(\boldsymbol{X}))\psi_\infty(\boldsymbol{x},\boldsymbol{X})\right)\\&+\frac{1}{2}\Delta_{\boldsymbol{X}}\psi_\infty(\boldsymbol{x},\boldsymbol{X}).\end{aligned} \tag{23}$$

This system is complemented with appropriate boundary conditions: for $\psi_\infty$ we impose the same boundary conditions with respect to $\boldsymbol{X}$ as on $\psi(t,.,.)$ (see above); and for $\boldsymbol{u}_\infty$ we impose the same Dirichlet boundary conditions as on $\boldsymbol{u}$ (or, as in Section 3.1, their limiting value when $t$ goes to infinity in case they are not constant in time).

We have not yet stated the precise boundary conditions on $\psi$ and $\psi_\infty$ with respect to the space variable $\boldsymbol{x}$. We simply assume that they are such that $\forall\boldsymbol{x}\in\partial\mathcal{D}$,

$$\boldsymbol{u}.\boldsymbol{\nu}\int_{\mathbf{R}^d}\psi\ln\left(\frac{\psi}{\psi_\infty}\right)=0, \tag{24}$$

where $\boldsymbol{\nu}$ denotes the outward normal to $\mathcal{D}$. This equality holds, for example, if a no penetration boundary condition is assumed on $\boldsymbol{u}$ ($\boldsymbol{u}.\boldsymbol{\nu}=0$ on $\partial\mathcal{D}$) and then there is no further condition on $\psi$. It also holds if the dumbbells are at equilibrium outside of $\mathcal{D}$ ($\psi=\psi_\infty$ on $\partial\mathcal{D}$).

Throughout this paper, we refer to the special stationary state corresponding to $\boldsymbol{u}_\infty=0$ as the *equilibrium* state. Such a state can, of course, only be reached with homogeneous (or vanishing in time) Dirichlet boundary conditions on the velocity.

In the equilibrium case, a natural stationary solution to the Fokker-Planck equation is:

$$\psi_\infty(\boldsymbol{X}) = \frac{\exp(-\Pi(\boldsymbol{X}))}{\int_{\mathbf{R}^d} \exp(-\Pi(\boldsymbol{X}))}. \tag{25}$$

Notice that $(\boldsymbol{u}_\infty, \psi_\infty)$ is then a solution to the system (20)–(21)–(22)–(23) (with adequate boundary conditions on $\psi_\infty$ and homogeneous Dirichlet boundary conditions on $\boldsymbol{u}_\infty$).

**Notation:** *Throughout this paper, $t \in \mathbb{R}_+$, $\boldsymbol{x} \in \mathcal{D}$ and $\omega \in \Omega$ denote the variable in time, space and the underlying probability space, respectively. Moreover, we only consider global-in-time $L^p_t$ spaces. For example, $\boldsymbol{X}_t(\boldsymbol{x}) \in L^2_t(L^2_{\boldsymbol{x}}(L^2_\omega))$ means that $\boldsymbol{X}_t$ is a measurable function of $(t, \boldsymbol{x}, \omega)$ and that $\|\boldsymbol{X}_t\|^2_{L^2_t(L^2_{\boldsymbol{x}}(L^2_\omega))} = \int_{\mathbf{R}_+} \int_{\mathcal{D}} \mathbb{E}(|\boldsymbol{X}_t(\boldsymbol{x})|^2)\, d\boldsymbol{x}\, dt < \infty$. In addition, for any vector $\boldsymbol{X}$, $|\boldsymbol{X}|$ denotes the Euclidean norm of $\boldsymbol{X}$, and for any matrix $\boldsymbol{\kappa}$, $|\boldsymbol{\kappa}|$ denotes the norm defined by $|\boldsymbol{\kappa}| = \sup_{|\boldsymbol{X}|=1} |\boldsymbol{\kappa}\boldsymbol{X}|$. We denote by $|\mathcal{D}|$ the Lebesgue measure of the domain $\mathcal{D}$. For any matrix $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}^T$ denotes the transposed matrix. Finally, we denote by $\mathcal{B}(0, \rho) \subset \mathbb{R}^d$ the ball centred at* $0$ *with radius* $\rho > 0$.

### *1.2. The case of a shear flow*

A simple setting, considered in Sections 3.1 and 3.3.2, is the coupled system (14)–(15)–(16)–(17) in a 2-dimensional plane shear flow geometry (see Fig. 2). In this case, $\boldsymbol{u}(t, \boldsymbol{x}) = (u(t, y), 0)$, where $\boldsymbol{x} = (0, y)$ and all the unknown fields depend only on $y$. We assume[2] that $y \in \mathcal{D} = (0, 1)$. System (14)–(15)–(16)–(17) may be written as:

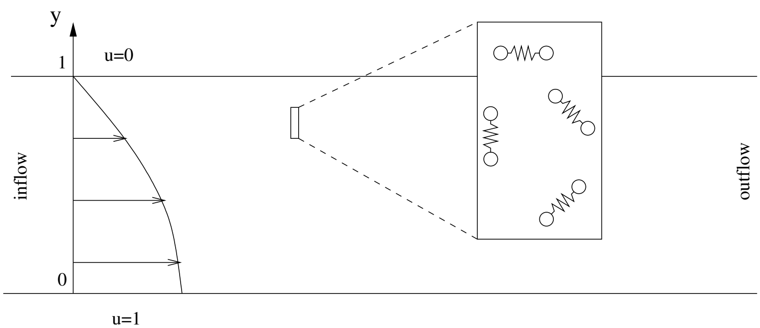


**Fig. 2.** Velocity profile in a shear flow of a dilute solution of polymers.

[2] The domain $\mathcal{D}$ denotes, in this case, the domain where the second component $y$ of $\boldsymbol{x}$ varies, and is therefore bounded and regular.

$$\frac{\partial u}{\partial t}(t, y) = \frac{\partial^2 u}{\partial y^2}(t, y) + \frac{\partial \tau}{\partial y}(t, y), \tag{26}$$

$$\tau(t, y) = \int_{\mathbf{R}^2} \left( X \frac{\partial \Pi}{\partial Y}(\boldsymbol{X}) \right) \psi(t, y, \boldsymbol{X})\, d\boldsymbol{X}, \tag{27}$$

$$\begin{aligned} \frac{\partial \psi}{\partial t}(t, y, \boldsymbol{X}) = & -\frac{\partial}{\partial X} \left( \left( \frac{\partial u}{\partial y}(t, y) Y - \frac{1}{2} \frac{\partial \Pi}{\partial X}(\boldsymbol{X}) \right) \psi(t, y, \boldsymbol{X}) \right) \\ & - \frac{\partial}{\partial Y} \left( -\frac{1}{2} \frac{\partial \Pi}{\partial Y}(\boldsymbol{X}) \psi(t, y, \boldsymbol{X}) \right) + \frac{1}{2} \Delta_{\boldsymbol{X}} \psi(t, y, \boldsymbol{X}), \end{aligned} \tag{28}$$

where $\boldsymbol{X} = (X, Y)$ and $\tau$ denotes the $(x, y)$ component of the stress tensor $\boldsymbol{\tau}$. The stochastic form of (27)–(28) is:

$$\tau(t, y) = \mathbf{E} \left( X_t(y) \frac{\partial \Pi}{\partial Y}(X_t(y), Y_t(y)) \right), \tag{29}$$

$$dX_t(y) = \left( \frac{\partial u}{\partial y}(t, y) Y_t(y) - \frac{1}{2} \frac{\partial \Pi}{\partial X}(X_t(y), Y_t(y)) \right) dt + dV_t, \tag{30}$$

$$dY_t(y) = -\frac{1}{2} \frac{\partial \Pi}{\partial Y}(X_t(y), Y_t(y))\, dt + dW_t, \tag{31}$$

where $\boldsymbol{X}_t(\boldsymbol{x}) = (X_t(y), Y_t(y))$ and $\boldsymbol{W}_t = (V_t, W_t)$.

One point of interest in this particular setting is that a stationary solution such that $u_\infty$ is affine, and $\psi_\infty$ does not depend on space, can be considered, so that the equations on $u_\infty$ and $\psi_\infty$ are decoupled (it is actually an example where $\boldsymbol{u}_\infty$ can be defined as a homogeneous flow (i.e., with a constant $\nabla \boldsymbol{u}_\infty$): see Section 3.3.2 for more generality).

In addition, in the particular case of Hookean dumbbells, the second component $Y_t$ does not depend on space and evolves independently of $(u, \tau, X_t)$. This particular case of Hookean dumbbells in a shear flow enables us to perform some specific computations, which we cannot generalize to a nonlinear force $\nabla \Pi$ such as the FENE force, or to another geometry. However, in Section 3.1 and Section 3.3.2 we present some results in this special case, since we believe the results to be of interest.

### *1.3. Outline of the paper and summary of the results*

In this paper we present some results of exponential long-time convergence towards a stationary state for the velocity $\boldsymbol{u}$ and the stress $\boldsymbol{\tau}$. As a preliminary, we begin in Section 2 by first considering the long-time behavior of the stress tensor obtained by (16)–(17) and (18)–(19), for a given vanishing or constant velocity field. In Section 3 we generalize the methods of Section 2 to deal with the coupled problem. First, we consider in Section 3.1 the long-time behavior of the coupled PDE-SDE system (26)–(29)–(30)–(31) for Hookean dumbbells in a shear flow. We show the exponential convergence of $u$ and $\tau$ towards some stationary state, under the assumption that the boundary conditions on $u$ converge exponentially fast to their limiting values (see Lemmas 2 and 3)[3]. We then turn to the long-time behavior of the coupled system of PDEs (14)–(15)–(16)–(17), in a general geometry, and for Hookean or FENE dumbbells. First, we consider the case of homogeneous Dirichlet boundary conditions on $\boldsymbol{u}$ in Section 3.2, and then in Section 3.3, the case of non-homogeneous Dirichlet boundary conditions on $\boldsymbol{u}$. In the case of homogeneous Dirichlet boundary conditions on $\boldsymbol{u}$, a classical entropy method enables us to show the exponential convergence of the velocity $\boldsymbol{u}$ and the density $\psi$ to their equilibrium values (see Proposition 5). From this we can deduce an exponential convergence of the stress tensor for Hookean dumbbells (see Proposition 7), but only a weak convergence of the stress tensor for FENE dumbbells (see Proposition 6). The situation is more intricate in the case of nonhomogeneous Dirichlet boundary conditions on the velocity $\boldsymbol{u}$. We have obtained convergence results for $(\boldsymbol{u}, \psi)$, but we have not been able to deduce from them any convergence result for the stress tensor. Moreover, we have only obtained complete results in the case of a homogeneous stationary flow (i.e., if $\boldsymbol{u}_\infty(\boldsymbol{x}) = \nabla \boldsymbol{u}_\infty \boldsymbol{x})$). If $\nabla \boldsymbol{u}_\infty$ is antisymmetric, the results regarding the exponential convergence of $\boldsymbol{u}$ and $\psi$ are the same as for homogeneous Dirichlet boundary conditions on $\boldsymbol{u}$. For a general $\nabla \boldsymbol{u}_\infty$, for FENE dumbbells and a sufficiently small homogeneous stationary flow, it is possible to prove the exponential convergence of $\boldsymbol{u}$ and $\psi$ (see Theorem 2), but we have not been able to obtain the same result for Hookean dumbbells, nor in the case of a nonhomogeneous stationary flow. In Table 1 we present a summary of the main results obtained.

We would like to emphasize that our arguments are partially *formal*: we assume that there exists a unique global-in-time solution regular enough so that the computations are valid. Moreover, we assume that the density of the process $\boldsymbol{X}_t$, which is a solution to (19), is regular enough so that it is the classical solution to the Fokker-Planck equation (17). Concerning the stationary state $(\boldsymbol{u}_\infty, \psi_\infty)$, three different situations arise:

(i) either we define $\boldsymbol{u}_\infty$ and $\psi_\infty$ explicitly, satisfying (20)–(21)–(22)–(23);
(ii) or we define $\boldsymbol{u}_\infty$ as an homogeneous flow (i.e., with a constant $\nabla \boldsymbol{u}_\infty$), compatible with the boundary conditions on the velocity, and we then define $\psi_\infty$ as a solution to (23) which does not depend on $\boldsymbol{x}$, so that $(\boldsymbol{u}_\infty, \psi_\infty)$ satisfies (20)–(21)–(22)–(23);
(iii) or we have no explicit expressions for any stationary state $(\boldsymbol{u}_\infty, \psi_\infty)$ so that $(\boldsymbol{u}_\infty, \psi_\infty)$ is defined as a solution to (20)–(21)–(22)–(23).

In any case, we again assume that $(\boldsymbol{u}_\infty, \psi_\infty)$ are sufficiently regular, and thus may require some bounds on some norms of these functions in order to obtain the results of long-time convergence of $(\boldsymbol{u}, \psi)$ to $(\boldsymbol{u}_\infty, \psi_\infty)$ (see in particular (A.126)). Note that we do not require *a priori* any uniqueness result on $(\boldsymbol{u}_\infty, \psi_\infty)$, but the uniqueness of a regular stationary state $(\boldsymbol{u}_\infty, \psi_\infty)$ to (20)–(21)–(22)–(23) will follow *a posteriori* from our derivations. With this limitation in mind, our aim is to prove the convergence to a stationary state under adequate assumptions.

[3] Of course, if the convergence of the boundary conditions on $u$ to their limiting values is slower, then it slows down the rate of convergence of $u$ and $\tau$ to their stationary state.

In this article, we alternate between some probabilistic proofs based on coupling methods associated with the system coupling the PDE with the SDE (14)–(15)–(18)–(19), and some analytic proofs based on the coupled system of PDEs (14)–(15)–(16)–(17). Even if the results we obtain by the probabilistic approaches are less general than the results obtained by using some analytic proofs, we think they are interesting since they illustrate how the basic assumptions required to obtain exponential convergence to equilibrium (such as the $\alpha$-convexity of $\Pi$) are used differently in the two approaches. Moreover, we think that the probabilistic proofs, which use only the process $\boldsymbol{X}_t$, are likely to be generalizable to the study of the long-time behavior of the discretized problem (by a Monte Carlo method, see the CONNFFESSIT method [27]), whereas the analytic proofs use nonlinear functionals of the density of $\boldsymbol{X}_t$ which cannot, in general, be easily expressed in terms of $\boldsymbol{X}_t$.

Let us recall what is known in the mathematical literature about these micro-macro models of polymeric fluids and the long-time behavior of kinetic models.

Existence results for micro-macro models of polymeric fluids are usually limited to small-time existence and uniqueness of strong solutions (except in the very special case of Hookean dumbbells in a shear flow [19]). In regards to the coupled PDE-SDE system (14)–(15)–(18)–(19) we refer to [20] (FENE model in shear flow, $b > 2$ or $b > 6$), [13] (for a polynomial force) and [33] (FENE model, $b > 76$ in dimension d=3). With regards to PDE coupled system (14)–(15)–(16)–(17), we refer the reader to the earlier work [29] (not for FENE), and also [21] (for a polynomial force). However, in a recent work [4], the existence of a global weak solution to the coupled PDE system for the FENE model with $b \geqq 10$ was proved, with a smoothing operator applied both on the velocity in (17) and on the stress tensor in (14).

Many methods to analyze the long-time behavior of kinetic models have been described, especially methods devoted to the analysis of the Boltzmann equation (which is much more complicated than the Fokker-Planck equation (17)). We would like to mention the review [14] on entropy methods for PDEs, and also [24, 10, 2]. See also [3] for a discretization in time which preserves the exponential convergence towards equilibrium. In many of these works on the Fokker-Planck equation, it is assumed that the stationary state satisfies the detailed balance (see Section 2.1 for a definition), or at least that $\psi_\infty$ is explicitly known, while in our framework, this is not necessarily the case (in particular if the stationary flow is not zero, see Section 3.3). Probabilistic methods used to study the long-time behavior of such systems (coupling methods) are described in [1] and also in [22, 25, 26].

Discussion of the long-time behavior of micro-macro models of polymeric fluids can be found in [12] (which contains some remarks on the long-time behavior of the decoupled system for FENE dumbbell). Many works are devoted to this question for liquid-crystal polymers: see [8, 7, 23] (decoupled system) and [28] (coupled system).

## 2. The decoupled case: long-time behavior for a given velocity field

In this section, we consider that the velocity field is known and is regular enough so that the vector field $\boldsymbol{u}(t, \boldsymbol{x})$ can be integrated, i.e., there exists a unique solution $\boldsymbol{x}(t, \boldsymbol{x}_0)$ to:

$$\begin{cases} \frac{d\boldsymbol{x}(t)}{dt} &= \boldsymbol{u}(t, \boldsymbol{x}(t)), \\ \boldsymbol{x}(0) &= \boldsymbol{x}_0. \end{cases}$$

It is then easy to check that for a given $\boldsymbol{x}_0$, if $\psi$ and $\boldsymbol{X}_t$ satisfy (17) and (19) respectively, then $\tilde{\psi}(t, \boldsymbol{X}) = \psi(t, \boldsymbol{x}(t, \boldsymbol{x}_0), \boldsymbol{X})$ and $\tilde{\boldsymbol{X}}_t = \boldsymbol{X}_t(\boldsymbol{x}(t, \boldsymbol{x}_0))$, which satisfy:

$$\frac{\partial \tilde{\psi}}{\partial t}(t, \boldsymbol{X}) = -\operatorname{div}\left(\left(\boldsymbol{G}(t)\boldsymbol{X} - \frac{1}{2}\nabla\Pi(\boldsymbol{X})\right)\tilde{\psi}(t, \boldsymbol{X})\right) + \frac{1}{2}\Delta\tilde{\psi}(t, \boldsymbol{X}), \quad (32)$$

$$d\tilde{\boldsymbol{X}}_t = \left(\boldsymbol{G}(t)\tilde{\boldsymbol{X}}_t - \frac{1}{2}\nabla\Pi(\tilde{\boldsymbol{X}}_t)\right) dt + d\boldsymbol{W}_t, \quad (33)$$

where

$$\boldsymbol{G}(t) = \nabla \boldsymbol{u}(t, \boldsymbol{x}(t, \boldsymbol{x}_0)) \quad (34)$$

and with initial condition $\tilde{\psi}(0, \boldsymbol{X}) = \psi(0, \boldsymbol{x}_0, \boldsymbol{X})$ and $\tilde{\boldsymbol{X}}_0 = \boldsymbol{X}_0(\boldsymbol{x}_0)$. We fix $\boldsymbol{x}_0$ and omit the *tilde* in the rest of this section. Note that the same Brownian motion $\boldsymbol{W}_t$ drives $\boldsymbol{X}_t$ and $\tilde{\boldsymbol{X}}_t$ since $\boldsymbol{W}_t$ does not depend on the space variable $\boldsymbol{x}$ (see [16] for a discussion of the modelling, mathematical and numerical issues raised by the Brownian motion dependence $\boldsymbol{x}$).

**Remark 2** (homogeneous flows)**.** If the boundary conditions on the velocity in (14)–(15)–(16)–(17) are such that $\boldsymbol{u}$ is a so-called homogeneous flow (i.e., there exists a tensor $\boldsymbol{\kappa}(t)$ such that $\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{\kappa}(t)\boldsymbol{x}$), it is natural to consider a solution $\psi$ (resp. $\boldsymbol{X}_t$) to (17) (resp. to (19)) which does not depend on space. Classical examples of homogeneous flows (see, for example, Chapter 3 in [5]) are shear flows for which $\boldsymbol{\kappa}(t) = \begin{bmatrix} 0 & \dot{\gamma}(t) \\ 0 & 0 \end{bmatrix}$, and elongational flows for which $\boldsymbol{\kappa}(t) = \begin{bmatrix} \dot{\varepsilon}(t) & 0 & 0 \\ 0 & m\dot{\varepsilon}(t) & 0 \\ 0 & 0 & -(1+m)\dot{\varepsilon}(t) \end{bmatrix}$, typically with $m \in \{-0.5, 0, 1\}$.

If $\psi$ or $\boldsymbol{X}_t$ are not space dependent, then the stress tensor $\boldsymbol{\tau}$ does not depend on space either, hence $\boldsymbol{\tau}$ is divergence free and $(\boldsymbol{u}, \psi)$ (resp. $(\boldsymbol{u}, \boldsymbol{X}_t)$) is a solution to (14)–(15)–(16)–(17) (resp. to (14)–(15)–(18)–(19)). Therefore, if the boundary conditions on the velocity are compatible with a homogeneous flow $\boldsymbol{u}$, it is natural to assume that the velocity field is given independently of $\psi$ or $\boldsymbol{X}_t$.

We are especially interested in the long-time behavior of the stress tensor $\boldsymbol{\tau}$ defined either by (16) or (18). In Section 2.1, we recall how entropy methods allow some information to be obtained on the long-time behavior of $\psi$, which is a solution to (32). We then show how the long-time behavior of $\boldsymbol{\tau}$ can be deduced from that of $\psi$ (Section 2.2), or studied using only the SDE (33) satisfied by $\boldsymbol{X}_t$, and not the density $\psi$ (Section 2.3).

The aim of Section 2 is to introduce the methods that will be used in the coupled case (namely the entropy methods, and the coupling methods), and to illustrate the role of the $\alpha$-convexity of $\Pi$.

### *2.1. Entropy methods for the Fokker-Planck equation: convergence for a constant* $\nabla \boldsymbol{u}$

In this section, we consider the special case of a steady homogeneous velocity field ($\boldsymbol{G}(t) = \boldsymbol{\kappa}$ is constant) and we recall some classical analytic methods based on the Fokker-Planck equation (32) to show the exponential convergence of $(\psi - \psi_\infty)$ to 0, where $\psi_\infty$ is defined as a stationary solution to (32). This section is based on the work of [2].

**2.1.1. Definition of the relative entropy.** A natural tool to analyze the long-time behavior of $\psi$ solution to (32) is the relative entropy

$$H(t) = \int_{\mathbf{R}^d} h\left(\frac{\psi(t, \boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right) \psi_\infty(\boldsymbol{X})\, d\boldsymbol{X}, \tag{35}$$

where $h : \mathbb{R} \to \mathbb{R}_+$ is a strictly convex $\mathcal{C}^2$ function, such that $h(1) = 0$ (see [2]). The following functions are typical examples of entropy functions:

$$h(x) = x \ln(x) - (x - 1), \tag{36}$$

$$h(x) = x^p - 1 - p(x - 1) \text{ with } 1 < p \leqq 2. \tag{37}$$

Notice that since by convexity $h(x) \geqq h'(1)(x - 1)$ with equality if, and only if, $x = 1$, $H(t)$ is nonnegative and, if $\psi(t, .) \neq \psi_\infty$, then positive. Using (32) and taking $\psi_\infty$ as a stationary solution to (32), $H$ can be shown to satisfy:

$$\frac{dH}{dt} = -\frac{1}{2} \int_{\mathbf{R}^d} \left|\nabla\left(\frac{\psi}{\psi_\infty}\right)\right|^2 h''\left(\frac{\psi}{\psi_\infty}\right) \psi_\infty, \tag{38}$$

and is thus a decreasing function. Notice that (38) only requires $\psi_\infty$ to be a stationary solution of (32): no explicit expression for $\psi_\infty$ is needed. For a rigorous derivation of (38), see Appendices A and B.

As soon as a functional inequality of the type: $\forall \phi$ such that $\phi \geqq 0$ and $\int_{\mathbf{R}^d} \phi = 1$,

$$\int_{\mathbf{R}^d} h\left(\frac{\phi}{\psi_\infty}\right) \psi_\infty \leqq C \int_{\mathbf{R}^d} \left|\nabla\left(\frac{\phi}{\psi_\infty}\right)\right|^2 h''\left(\frac{\phi}{\psi_\infty}\right) \psi_\infty, \tag{39}$$

holds for some constant $C$ (which of course depends on $\psi_\infty$), (38) then implies:

$$H(t) \leqq H(0) \exp\left(-\frac{t}{2C}\right), \tag{40}$$

with the same constant $C$, thereby showing the exponential decay of $H$ to zero, which is a measure of the convergence of $\psi$ to $\psi_\infty$. More precisely, in the special case $h(x) = x\ln(x) - (x-1)$, using then the Csiszar-Kullback inequality: $\forall\phi, \psi_\infty$ such that $\phi, \psi_\infty \geqq 0$ and $\int_{\mathbf{R}^d}\phi = \int_{\mathbf{R}^d}\psi_\infty = 1$,

$$\left(\int_{\mathbf{R}^d} |\phi - \psi_\infty|\right)^2 \leqq 4\int_{\mathbf{R}^d} \phi \ln\left(\frac{\phi}{\psi_\infty}\right), \tag{41}$$

we obtain the convergence in the $L^1_{\boldsymbol{X}}$ norm of $\psi(t, \boldsymbol{X})$ to $\psi_\infty(\boldsymbol{X})$ at an exponential rate. Actually, as soon as $h$ is a strictly convex function, such that $h(1) = 0$ and $\left(\frac{1}{h''}\right)'' \leqq 0$ (which is true for (36) and (37) and will be a natural requirement, see below), an inequality: $\forall\phi, \psi_\infty$ such that $\phi, \psi_\infty \geqq 0$ and $\int_{\mathbf{R}^d}\phi = \int_{\mathbf{R}^d}\psi_\infty = 1$,

$$\left(\int_{\mathbf{R}^d} |\phi - \psi_\infty|\right)^2 \leqq C\int_{\mathbf{R}^d} h\left(\frac{\phi}{\psi_\infty}\right)\psi_\infty$$

holds for some positive constant $C$ (see Equation (2.26) in [2]). We will see in Section 2.2 (see Propositions 2 and 3), how the exponential convergence of the stress tensor to its equilibrium value for Hookean or FENE dumbbells can be deduced from the exponential convergence in the $L^1_{\boldsymbol{X}}$ norm of $\psi(t, \boldsymbol{X})$ to $\psi_\infty(\boldsymbol{X})$.

Therefore, we see that the crucial ingredient to prove the exponential decay (40) of $H$ to 0 is the functional inequality (39). Let us now consider two cases: $\boldsymbol{\kappa} = 0$ and $\boldsymbol{\kappa} \neq 0$.

**2.1.2. The case $\boldsymbol{\kappa} = 0$ (a direct proof of (40)).** If $\boldsymbol{\kappa} = 0$, then, as already mentioned in the introduction, $\psi_\infty \propto \exp(-\Pi)$ is a stationary solution to (32). In this case, $\psi_\infty$ satisfies the detailed balance in the sense that it solves not only (32) but indeed:

$$\left(-\boldsymbol{\kappa}\boldsymbol{X} + \frac{1}{2}\nabla\Pi(\boldsymbol{X})\right)\psi_\infty(\boldsymbol{X}) + \frac{1}{2}\nabla\psi_\infty(\boldsymbol{X}) = 0. \tag{42}$$

In this case, one way to prove (39) is to first prove (40) for any initial condition $\psi(0, .)$, choose $\psi(0, .) = \phi$, and then consider an expansion of $H$ around 0.

**Remark 3.** The functional inequality (39) is actually equivalent to the fact that the inequality (40) holds for any initial condition $\psi(0, \boldsymbol{X})$. To be more precise, the fact that (39) holds for any density $\phi$ is equivalent to the fact that the inequality (40) holds for any initial condition $\psi(0, .) = \phi$, where $H(t)$ is defined by (35), with $\psi$ being a solution of (32) with $\boldsymbol{G} = 0$ and $\Pi = -\ln(\psi_\infty)$.

In [2] (see Theorem 2.16), it is proved that (40) holds under the following assumptions:

(i) $\left(\frac{1}{h''}\right)'' \leqq 0$ (which we assume henceforth, and which is true for (36) and (37)),
(ii) $\Pi$ is $\alpha$-convex,
(iii) $\psi_\infty$ satisfies the detailed balance (42).

The proof consists in computing $\frac{d^2H}{dt^2}$ and uses, in particular, the fact that $H(\psi(t))$ converges to $H(\psi_\infty)$ as time goes to infinity, a fact that can be proven independently under the same assumptions as above (see Lemma 2.11 in [2]). Therefore, we have proved that if $\boldsymbol{\kappa} = 0$, then $H(t)$ converges exponentially fast to 0. This implies the following well-known result: for any density $\psi_\infty$, if $-\ln(\psi_\infty)$ is $\alpha$-convex, then (39) holds for $\psi_\infty$. More precisely, if $-\ln(\psi_\infty)$ is $\alpha$-convex, (39) holds with a positive constant $C$ such that (see [2] Corollary 2.18):

$$C \leqq \frac{1}{2\alpha}. \tag{43}$$

**2.1.3. The case $\boldsymbol{\kappa} \neq 0$ (a perturbation result to prove (39)).** Let us now consider the case of a constant non-zero $\boldsymbol{\kappa}$. Compared to the case $\boldsymbol{\kappa} = 0$, (38) still holds, but $\psi_\infty$ may no longer satisfy (42), nor be explicitly known. Then, the natural way to prove (40) is to first prove (39). The fact that $-\ln(\psi_\infty)$ is $\alpha$-convex is a sufficient, but not necessary, condition for (39) to hold. In the case when $-\ln(\psi_\infty)$ may not be $\alpha$-convex, a perturbation result can be used to prove (39). Let us concentrate on two functions $h$: (36), and (37) for $p = 2$. In the case $h(x) = x\ln(x) - (x-1)$, (39) is called a logarithmic Sobolev inequality with respect to $\psi_\infty$ (and we denote by $C_{\mathrm{LSI}}(\psi_\infty)$ the constant $C$ for which (39) holds). In the case $h(x) = x^2 - 1 - 2(x-1)$, (39) is called a Poincaré inequality with respect to $\psi_\infty$ (and we denote by $C_{\mathrm{PI}}(\psi_\infty)$ the constant $C$ for which (39) holds). In both of these cases, we have the following perturbation result (see [1] Theorem 3.4.1 p. 49 and Theorem 3.4.3 p. 50, see also Theorem 3.2 in [2] for a generalization to any function $h$):

**Lemma 1.** *If a logarithmic Sobolev inequality (resp. a Poincaré inequality) holds for $\psi_\infty \propto \exp(-\Pi)$ with a constant $C_{\mathrm{LSI}}(\psi_\infty)$ (resp. $C_{\mathrm{PI}}(\psi_\infty)$) and if $\tilde{\Pi}$ is a bounded function, then a logarithmic Sobolev inequality (resp. a Poincaré inequality) holds for the density $\widetilde{\psi_\infty} \propto \exp(-\Pi + \tilde{\Pi})$ with a constant $C_{\mathrm{LSI}}(\widetilde{\psi_\infty}) \leqq C_{\mathrm{LSI}}(\psi_\infty)\exp(2\operatorname{osc}(\tilde{\Pi}))$ (resp. a constant $C_{\mathrm{PI}}(\widetilde{\psi_\infty}) \leqq C_{\mathrm{PI}}(\psi_\infty)\exp(2\operatorname{osc}(\tilde{\Pi}))$), where*

$$\operatorname{osc}(\tilde{\Pi}) = \sup(\tilde{\Pi}) - \inf(\tilde{\Pi}) \tag{44}$$

*denotes the oscillation of $\tilde{\Pi}$.*

It is now shown how this result can be used to prove the exponential decay of $H$ to 0 in the case of a constant $\boldsymbol{G} = \boldsymbol{\kappa} \neq 0$. As already mentioned, (38) still holds, but the point of interest is (39). There are three situations where we are able to prove (39) and therefore prove the exponential decay of $H$ to 0:

(i) The tensor $\boldsymbol{\kappa}$ is antisymmetric, $\Pi$ is $\alpha$-convex and we choose $\psi_\infty \propto \exp(-\Pi)$ as a stationary solution to (32). Then, by the $\alpha$-convexity of $-\ln(\psi_\infty)$, (39) holds. Therefore, one obtains the exponential decay of $H$ to 0. Note that in this case, $\psi_\infty$ does not satisfy the detailed balance (42).

(ii) The tensor $\boldsymbol{\kappa}$ is symmetric, $\Pi$ is
  (a) the FENE potential,
  (b) or the Hookean potential, and then we assume that the eigenvalues of $\boldsymbol{\kappa}$ are strictly smaller than $1/2$,

and we choose $\psi_\infty(\boldsymbol{X}) \propto \exp(-\Pi(\boldsymbol{X}) + \boldsymbol{X}.\boldsymbol{\kappa}\boldsymbol{X})$ as a stationary solution of (32) (in this case, $\psi_\infty$ satisfies (42)). In the case of FENE dumbbells, from Lemma 1, (39) is obtained since $\operatorname{osc}\left(\boldsymbol{X}.\boldsymbol{\kappa}\boldsymbol{X}1_{|\boldsymbol{X}|<\sqrt{b}}\right) < \infty$. Therefore, the exponential decay of $H(t)$ to zero for FENE dumbbells and for any symmetric $\boldsymbol{\kappa}$ has been proved. For Hookean dumbbells, (39) holds for some constant $C$ as long as $\int \exp(-\Pi(\boldsymbol{X}) + \boldsymbol{X}.\boldsymbol{\kappa}\boldsymbol{X}) < \infty$. Indeed, this is equivalent to the fact that the eigenvalues of $\boldsymbol{\kappa}$ are strictly smaller than $1/2$, and implies that $-\Pi(\boldsymbol{X}) + \boldsymbol{X}.\boldsymbol{\kappa}\boldsymbol{X}$ is $\alpha$-convex.

(iii) The matrix $\boldsymbol{\kappa}$ is arbitrary, and $\psi_\infty$ is defined as a stationary solution to (32) which satisfies $\operatorname{osc}\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right) < \infty$. Then, by Lemma 1, (39) holds and hence $H$ decays exponentially fast to zero. It is then natural to question whether it is possible to build a stationary solution to (32) which satisfies $\operatorname{osc}\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right) < \infty$. This can actually be done in the FENE case, under the additional assumption that the symmetric part of $\boldsymbol{\kappa}$ is small enough (see Proposition 10 below).

Let us now summarize the results of this section:

**Proposition 1.** *If* $\psi$ *is a solution to* (32) *with* $\boldsymbol{G}(t) = \boldsymbol{\kappa} = 0$ *and* $\Pi$ *is a* $\alpha$*-convex potential, then the entropy* $H$ *defined by* (35)*, with* $\psi_\infty \propto \exp(-\Pi)$ *converges exponentially fast to 0.*

*Moreover, it has been shown that the exponential decay of* $H$ *to 0 also holds in the following cases:*

*(i) for any* $\alpha$*-convex potential, if* $\boldsymbol{\kappa}$ *is antisymmetric and* $\psi_\infty \propto \exp(-\Pi)$*,*
*(ii) for any* $\alpha$*-convex potential, any matrix* $\boldsymbol{\kappa}$*, and* $\psi_\infty$ *is a stationary solution to* (32) *such that* $\operatorname{osc}\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right) < \infty$*,*
*(iii) for FENE dumbbells, if* $\boldsymbol{\kappa}$ *is symmetric and* $\psi_\infty \propto \exp(-\Pi + \boldsymbol{X}.\boldsymbol{\kappa}\boldsymbol{X})$*,*
*(iv) for Hookean dumbbells, if* $\boldsymbol{\kappa}$ *is symmetric and has eigenvalues smaller than* $1/2$ *and* $\psi_\infty \propto \exp(-\Pi + \boldsymbol{X}.\boldsymbol{\kappa}\boldsymbol{X})$*,*
*(v) for FENE dumbbells, for any* $\boldsymbol{\kappa}$ *such that the symmetric part of* $\boldsymbol{\kappa}$ *is small enough, for a regular* $\psi_\infty$ *built in such a way that* $\operatorname{osc}\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right) < \infty$ *(see Proposition 10).*

*In all the above cases, by the Csiszar-Kullback inequality (41), the exponential convergence of the entropy* $H$ *implies the exponential convergence to 0 of the* $L^1_{\boldsymbol{X}}$ *norm of* $(\psi(t,.) - \psi_\infty)$*.*

**Remark 4.** The convergence of $H$ to 0 implies the uniqueness of a regular stationary state $\psi_\infty$, by uniqueness of the limit.

### *2.2. Long-time convergence of the stress tensor*

#### 2.2.1. Polynomial growth of $\boldsymbol{X}_t \otimes \nabla\Pi(\boldsymbol{X}_t)$.

We first show that under some assumptions on $\boldsymbol{G}(t)$, the growth in time of the $L^r_\omega$ norm ($1 < r < \infty$) of $\boldsymbol{X}_t \otimes \nabla\Pi(\boldsymbol{X}_t)$ is at most affine, for Hookean and FENE dumbbells.

**Proposition 2** (Polynomial growth of $X_t \otimes \nabla \Pi(X_t)$)**.** *In the case of Hookean dumbbells, if $G(t)$ is in $L^p_t$ for some $p \in [1,\infty)$, then, for all $r \in [1,\infty)$, $\exists C, M > 0$, $\forall t \geqq 0$,*

$$\left(\mathbf{E}\,|X_t \otimes \nabla \Pi(X_t)|^r\right)^{1/r} \leqq C + Mt. \tag{45}$$

*In the special case of a shear flow, for which $G(t) = \begin{bmatrix} 0 & V(t) \\ 0 & 0 \end{bmatrix}$, a sufficient condition is that $V \in L^p_t$ for some $p \in [1,\infty]$.*

*In the case of FENE dumbbells, if $G(t)$ is in $L^2_t + L^\infty_t$, then (45) holds for all $r \in (1, b/2-1)$ provided that there exists a $p > r$, s.t. $\mathbf{E}\left(\frac{1}{1-|X_0|^2/b}\right)^p < \infty$.*

**Proof.** For the Hookean case, since $\nabla \Pi(X) = X$, we only need to estimate $\mathbf{E}|X_t|^{2r}$, and since $X_t$ is Gaussian, it is enough to estimate $\mathbf{E}|X_t|^2$. The result then follows from Itô's calculus on $|X_t|^2$: $\forall t \geqq 0$,

$$\mathbf{E}|X_t|^2 \leqq \mathbf{E}|X_0|^2 + \int_0^t (2|G(s)| - 1)\mathbf{E}|X_s|^2\, ds + t\, d.$$

Therefore, $\mathbf{E}|X_t|^2 \leqq \mathbf{E}|X_0|^2 e^{\lambda(t)} + d \int_0^t e^{\lambda(t)-\lambda(s)}\, ds$ with $\lambda(t) = \int_0^t 2|G|(u)\, du - t$. The fact that $\sup_{0 \leqq s \leqq t < \infty}(\lambda(t) - \lambda(s)) < \infty$ if $G(t)$ is in $L^p_t$ for a $p \in [1,\infty)$ completes the proof. The special case of a shear flow can be treated in a straightforward manner.

For the FENE case, the proof is contained in Lemmas 2 and 3 of [20]. The proof is based on a Girsanov transform to treat the $L^2_t$ part of $G(t)$, and on an adaptation of the proof of Lemma 2 to deal with the $L^\infty_t$ part of $G(t)$. □

**2.2.2. Exponential long-time convergence of the stress tensor.** In Section 2.1 (see Proposition 1), we showed that if $G(t) = \kappa = 0$, $\psi(t, X)$ solution to (17) converges in the $L^1_X$ norm to $\psi_\infty(X)$ exponentially fast (for any $\alpha$-convex potential $\Pi$ and therefore for Hookean and FENE dumbbells). By Proposition 2, we know that for FENE or Hookean dumbbells if $G(t) = \kappa = 0$, then the $L^r_\omega$ norm $(1 < r < \infty)$ of $X \otimes \nabla \Pi(X)$ has a polynomial growth (under adequate assumptions on the initial condition and $b$). We now prove the exponential convergence of the stress tensor under these assumptions by using the following result:

**Proposition 3** (Exponential convergence of the stress tensor)**.** *It is assumed that $\psi(t, X)$ that is a solution of (17) converges in the $L^1_X$ norm to $\psi_\infty(X)$ exponentially fast (or equivalently that the law of $X_t$ that is a solution of (19) converges exponentially fast in variation to $\psi_\infty(X)\, dX$). In addition, it is assumed that there exists some $q > 1$ such that*

$$\int_{\mathbf{R}^d} |X \otimes \nabla \Pi(X)|^q \psi_\infty(X)\, dX < \infty,$$

*and that the growth in time of the $L^q_\omega$ norm of $X_t \otimes \nabla \Pi(X_t)$ is polynomial: there exists a polynomial $P$, $\forall t \geqq 0$,*

$$\mathbf{E}\,|X_t \otimes \nabla \Pi(X_t)|^q \leqq P(t).$$

*Then the stress tensor $\boldsymbol{\tau}(t)$ defined by (16) or (18) converges exponentially fast to $\boldsymbol{\tau}_\infty = \int_{\mathbf{R}^d} \boldsymbol{X} \otimes \nabla\Pi(\boldsymbol{X})\psi_\infty(\boldsymbol{X})\, d\boldsymbol{X}$.*

**Proof.** The proof simply results from the Hölder inequality:

$$
\begin{aligned}
&\left| \int_{\mathbf{R}^d} \boldsymbol{X} \otimes \nabla\Pi(\boldsymbol{X})(\psi(t,\boldsymbol{X}) - \psi_\infty(\boldsymbol{X})) \right| \\
&\leqq \int_{\mathbf{R}^d} |\boldsymbol{X} \otimes \nabla\Pi(\boldsymbol{X})|\, |\psi(t,\boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \\
&\leqq \left( \int_{\mathbf{R}^d} |\boldsymbol{X} \otimes \nabla\Pi(\boldsymbol{X})|^q \, |\psi(t,\boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^{1/q} \\
&\quad \left( \int_{\mathbf{R}^d} |\psi(t,\boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^{1/p} \\
&\leqq \left( \mathbf{E}\, |\boldsymbol{X}_t \otimes \nabla\Pi(\boldsymbol{X}_t)|^q + \int_{\mathbf{R}^d} |\boldsymbol{X} \otimes \nabla\Pi(\boldsymbol{X})|^q \psi_\infty(\boldsymbol{X}) \right)^{1/q} \\
&\quad \left( \int_{\mathbf{R}^d} |\psi(t,\boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^{1/p} .
\end{aligned}
$$

Since the first term in the right-hand side has polynomial growth with time, and the second term decays exponentially fast to zero, then $\boldsymbol{\tau}(t)$ converges exponentially fast to $\boldsymbol{\tau}_\infty$. □

Notice that by combining the results of Propositions 1, 2 and 3, we have also shown the convergence of the stress tensor for FENE dumbbells if, for example, $\boldsymbol{G}(t)$ is constant.

### *2.3. A coupling method for the SDE: convergence for a vanishing $\nabla \boldsymbol{u}$*

In this section, we want to briefly mention a coupling method to study the long-time behavior of $\boldsymbol{X}_t$ that is a solution to (33), which we will then extend to the coupled problem, at least in a simple case (see Section 3.1). Here we refer to [17]. We assume[4] that $|\boldsymbol{G}(t)| \leqq C \exp(-\alpha t)$. The idea is to introduce the stationary process $\boldsymbol{X}_t^\infty$ which is a solution to:

$$
d\boldsymbol{X}_t^\infty = \left( -\frac{1}{2} \nabla\Pi(\boldsymbol{X}_t^\infty) \right) dt + d\boldsymbol{W}_t, \tag{46}
$$

where $\boldsymbol{X}_0^\infty$ is a random variable independent from $\boldsymbol{W}_t$ with law $\exp(-\Pi(\boldsymbol{X}))\, d\boldsymbol{X}$. Note that the Brownian motion $\boldsymbol{W}_t$ is the same as in the SDE (33) satisfied by $\boldsymbol{X}_t$. By subtracting (46) from (33), we can then show the exponential long-time

[4] If the convergence of $|\boldsymbol{G}(t)|$ to zero is slower than exponential, our arguments show the convergence of $\|\boldsymbol{X}_t - \boldsymbol{X}_t^\infty\|_{L^k_\omega}$ to 0, but with a slower rate.

convergence of $\|X_t - X_t^\infty\|_{L^k_\omega}$ to 0 (for any $k \geqq 1$). The key inequality in this approach is:

$$(X_t - X_t^\infty).(\nabla\Pi(X_t) - \nabla\Pi(X_t^\infty)) \geqq \alpha|X_t - X_t^\infty|^2, \tag{47}$$

which is the $\alpha$-convexity of $\Pi$. We refer to [17] for further details and the proof of the convergence of the stress tensor by this approach.

**Remark 5.** Notice that by this simple coupling method, the convergence of the law of $X_t$ to $\exp(-\Pi(X))\,dX$ is obtained in the Wasserstein distance $W_k$ for $k \geqq 1$, but not in variation (i.e., for $k = 0$).

## 3. The coupled case

We are now in a position to study the long-time behavior of the coupled system (14)–(15)–(18)–(19) (in Section 3.1) or (14)–(15)–(16)–(17) (in Sections 3.2 and 3.3). We shall consider three settings. In Section 3.1, we use the coupling method introduced in Section 2.3 to prove the exponential convergence to equilibrium in the simple case of Hookean dumbbells in a shear flow. This method does not seem to apply to a more general framework. In Section 3.2, we employ the entropy method introduced in Section 2.1 to demonstrate the exponential convergence to equilibrium, in the case of homogeneous Dirichlet boundary conditions on the velocity $u$. We then consider in Section 3.3 the case of nonhomogeneous Dirichlet boundary conditions on $u$ and show how for the FENE model, an appropriate estimate on $\nabla\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right)$ allows completion of the proof. The latter holds for the FENE model under the additional assumption that $u_\infty$ is homogeneous (see Proposition 10).

### *3.1. Long-time behavior of the Hookean dumbbells model in a shear flow using a coupling method*

In this section, we consider the coupled system (26)–(29)–(30)–(31) for Hookean dumbbells (5) in a shear flow. The process $Y_t(y)$ is defined here independently of $(u(t, y), X_t(y))$, and there is only a coupling between $u(t, y)$ and $X_t(y)$. We impose the following initial conditions on the system (26)–(29)–(30)–(31): $u(0, y) = u_0(y)$ and $(X_0(y), Y_0(y))$. We assume that $(X_0(y))_{0\leqq y\leqq 1}$ and $(Y_0(y))_{0\leqq y\leqq 1}$ are independent random fields belonging to $L^2_y(L^2_\omega)$. In addition we assume that $(X_0(y), Y_0(y))_{0\leqq y\leqq 1}$ is independent of the two-dimensional Brownian motion $(V_t, W_t)$. We also add to this system Dirichlet boundary conditions on $u$: $u(t, 0) = f_0(t)$ and $u(t, 1) = f_1(t)$. We suppose that

$$\lim_{t\to\infty} f_0(t) = a_0 \qquad \lim_{t\to\infty} f_1(t) = a_1, \tag{48}$$

so that the asymptotic state for the velocity is expected to be

$$u_\infty(y) = a_0 + y(a_1 - a_0). \tag{49}$$

Correspondingly, we introduce the processes $(X_t^\infty, Y_t^\infty)$ which satisfy the following SDE:

$$dX_t^\infty = ((a_1 - a_0)Y_t^\infty - \frac{1}{2}X_t^\infty)\,dt + dV_t,$$
$$dY_t^\infty = -\frac{1}{2}Y_t^\infty\,dt + dW_t,$$

with initial condition which ensures that $(X_t^\infty, Y_t^\infty)$ is a stationary Gaussian process independent of $y$: $(X_0^\infty, Y_0^\infty)$ is independent of $(V_t, W_t)$ with law $\mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{bmatrix}1 + 2(a_1 - a_0)^2 & (a_1 - a_0)\\ (a_1 - a_0) & 1\end{bmatrix}\right)$. In addition, we choose $(X_0^\infty, Y_0^\infty)$ such that

$$(X_0^\infty, Y_0^\infty) \text{ is independent of } (Y_0(y))_{0\leq y\leq 1}. \tag{50}$$

The stress $\mathbf{E}(X_tY_t)$ is expected to converge to $\tau_\infty = \mathbf{E}(X_0^\infty Y_0^\infty) = a_1 - a_0$. The triple $(u_\infty, X^\infty, Y^\infty)$ is a solution to the following system:

$$0 = \frac{\partial^2 u_\infty}{\partial y^2} + \frac{\partial \tau_\infty}{\partial y},$$
$$\tau_\infty = \mathbf{E}\left(X_t^\infty Y_t^\infty\right),$$
$$dX_t^\infty = \frac{\partial u_\infty}{\partial y}Y_t^\infty - \frac{1}{2}X_t^\infty\,dt + dV_t,$$
$$dY_t^\infty = -\frac{1}{2}Y_t^\infty\,dt + dW_t,$$

with boundary conditions $u_\infty(0) = a_0$, $u_\infty(1) = a_1$, and where $\tau_\infty$ does not depend on time and space, since $(X^\infty, Y^\infty)$ is a stationary process not depending on space. We are able to prove the convergence to the stationary state $(u_\infty, \tau_\infty)$ where $u_\infty$ is defined by (49) and $\tau_\infty = a_1 - a_0$:

**Lemma 2.** *Owing to the very nature of the above system, $\|Y_t(y) - Y_t^\infty\|_{L^2_y(L^2_\omega)}$ converges exponentially fast to zero. It is assumed, in addition to (48)–(50), that $f_0, f_1 \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}_+)$ and*

$$\lim_{t\to\infty} \dot{f}_0(t) = \lim_{t\to\infty} \dot{f}_1(t) = 0, \tag{51}$$

*where $\dot{f}$ denotes the derivative of $f$ with respect to time; $(u, X)$ will then converge to $(u_\infty, X^\infty)$ as $t \longrightarrow \infty$ in the following sense:*

$$\lim_{t\to\infty} \|u(t, y) - u_\infty(y)\|_{L^2_y} = 0,$$
$$\lim_{t\to\infty} \|X_t(y) - X_t^\infty\|_{L^2_y(L^2_\omega)} = 0.$$

*In particular, the stress $\mathbf{E}(X_t(y)Y_t(y))$ converges to $\tau_\infty = a_1 - a_0$ in $L^1_y$ as $t \to \infty$.*

**Proof.** In this proof, $C$ denotes various positive constants independent of the data.

Using the explicit expressions of $Y_t(y)$ and $Y_t^\infty$, it is easy to check that

$$\|Y_t(y) - Y_t^\infty\|_{L^2_y(L^2_\omega)} = \|Y_0(y) - Y_0^\infty\|_{L^2_y(L^2_\omega)} e^{-t/2}. \tag{52}$$

We introduce

$$\tilde{u}(t, y) = f_0(t) + y(f_1(t) - f_0(t)),$$

and $\tilde{X}_t$, defined by,

$$\tilde{X}_t = X_0^\infty + \int_0^t \left( (f_1(s) - f_0(s)) Y_s^\infty - \frac{1}{2}\tilde{X}_s \right) ds + V_t.$$

Clearly, we have:

$$\|\tilde{u} - u_\infty\|_{L^2_y} \leqq C\left(|f_0(t) - a_0| + |f_1(t) - a_1|\right). \tag{53}$$

Moreover, it is easy to show that

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{X} - X^\infty\|^2_{L^2_\omega} \\
&\quad \leqq \left( 2(|f_0(t) - a_0| + |f_1(t) - a_1|)\|\tilde{X} - X^\infty\|_{L^2_\omega} - \|\tilde{X} - X^\infty\|^2_{L^2_\omega} \right),
\end{aligned}$$

so that

$$\frac{d}{dt}\|\tilde{X} - X^\infty\|_{L^2_\omega} \leqq \left( (|f_0(t) - a_0| + |f_1(t) - a_1|) - \frac{1}{2}\|\tilde{X} - X^\infty\|_{L^2_\omega} \right),$$

and

$$\|\tilde{X} - X^\infty\|_{L^2_\omega} \leqq \int_0^t (|f_0(s) - a_0| + |f_1(s) - a_1|) \exp(-(t - s)/2)\, ds. \tag{54}$$

Therefore by Lemma 3(ii) below, we obtain:

$$\lim_{t\to\infty} \|\tilde{X} - X^\infty\|_{L^2_\omega} = 0. \tag{55}$$

It now remains to compare $X$ with $\tilde{X}$, and $u$ with $\tilde{u}$. Notice that, since $\tilde{X}$ and $Y^\infty$ do not depend on space, $(u - \tilde{u})$ and $(X - \tilde{X})$ are solutions to:

$$\begin{aligned}
\frac{\partial(u - \tilde{u})}{\partial t}(t, y) &= \frac{\partial^2(u - \tilde{u})}{\partial y^2}(t, y) + \frac{\partial(\tau - \tilde{\tau})}{\partial y}(t, y) \\
&\quad - (\dot{f}_0(t) + y(\dot{f}_1(t) - \dot{f}_0(t))), && (56)\\
\tau(t, y) &= \mathbf{E}\left(X_t(y) Y_t(y)\right), && (57)\\
\tilde{\tau}(t) &= \mathbf{E}\left(\tilde{X}_t Y_t^\infty\right), && (58)\\
\frac{d(X_t(y) - \tilde{X}_t)}{dt} &= \left( \frac{\partial u}{\partial y}(t, y) Y_t(y) - \frac{\partial \tilde{u}}{\partial y}(t) Y_t^\infty \right) - \frac{1}{2}(X_t(y) - \tilde{X}_t). && (59)
\end{aligned}$$

We multiply (56) by $(u - \tilde{u})$ and (59) by $(X - \tilde{X})$. Using the fact that $(u - \tilde{u})$ is zero both for $y = 0$ and $y = 1$, and that $f_0$ and $f_1$ are bounded functions, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u-\tilde{u}\|^2_{L^2_y} + \|X_t - \tilde{X}_t\|^2_{L^2_y(L^2_\omega)}\right)\\
&\quad = -\left\|\frac{\partial(u-\tilde{u})}{\partial y}\right\|^2_{L^2_y} - \frac{1}{2}\|X_t - \tilde{X}_t\|^2_{L^2_y(L^2_\omega)}\\
&\qquad - \int (\dot{f}_0(t) + y(\dot{f}_1(t) - \dot{f}_0(t)))(u-\tilde{u})\\
&\qquad - \int \mathbf{E}\left(X_t(y)Y_t(y) - \tilde{X}_t Y_t^\infty\right)\frac{\partial(u-\tilde{u})}{\partial y}\\
&\qquad + \int \mathbf{E}\left((\frac{\partial u}{\partial y}(t,y)Y_t(y) - \frac{\partial \tilde{u}}{\partial y}(t)Y_t^\infty)(X_t(y) - \tilde{X}_t)\right),\\
&\quad \leqq C\left(-\|u-\tilde{u}\|^2_{L^2_y} - \|X_t - \tilde{X}_t\|^2_{L^2_y(L^2_\omega)} + (|\dot{f}_0| + |\dot{f}_1|)\|u-\tilde{u}\|_{L^2_y}\right)\\
&\qquad + \int \mathbf{E}\left((X_t(y) - \tilde{X}_t)(Y_t(y) - Y_t^\infty)\right)\frac{\partial \tilde{u}}{\partial y}(t)\\
&\qquad + \int \mathbf{E}\left(\tilde{X}_t(Y_t(y) - Y_t^\infty)\right)\frac{\partial(\tilde{u}-u)}{\partial y}(t,y)\\
&\quad \leqq C\left(-\|u-\tilde{u}\|^2_{L^2_y} - \|X_t - \tilde{X}_t\|^2_{L^2_y(L^2_\omega)} + (|\dot{f}_0| + |\dot{f}_1|)\|u-\tilde{u}\|_{L^2_y}\right.\\
&\qquad \left. + e^{-t/2}\|X_t - \tilde{X}_t\|_{L^2_y(L^2_\omega)}\right),
\end{aligned}$$

where we have also used the fact that,

$$\int \mathbf{E}\left(\tilde{X}_t(Y_t(y) - Y_t^\infty)\right)\frac{\partial(\tilde{u}-u)}{\partial y}(t,y) = 0.$$

The above equation holds since $\mathbf{E}\left(\tilde{X}_t(Y_t(y) - Y_t^\infty)\right)$ does not depend on $y$, which can be checked by using (50) and the explicit expressions of $Y_t$, $Y_t^\infty$ and $\tilde{X}_t$. Therefore,

$$\begin{aligned}
&\frac{d}{dt}\left(\|u-\tilde{u}\|^2_{L^2_y} + \|X_t - \tilde{X}_t\|^2_{L^2_y(L^2_\omega)}\right)\\
&\quad \leqq C\Big(-\|u-\tilde{u}\|^2_{L^2_y} - \|X_t - \tilde{X}_t\|^2_{L^2_y(L^2_\omega)}\\
&\qquad + (|\dot{f}_0| + |\dot{f}_1| + e^{-t/2})(\|u-\tilde{u}\|_{L^2_y} + \|X_t - \tilde{X}_t\|_{L^2_y(L^2_\omega)})\Big)\\
&\quad \leqq C\Bigg(-\|u-\tilde{u}\|^2_{L^2_y} - \|X_t - \tilde{X}_t\|^2_{L^2_y(L^2_\omega)}\\
&\qquad + \sqrt{2}(|\dot{f}_0| + |\dot{f}_1| + e^{-t/2})\sqrt{\|u-\tilde{u}\|^2_{L^2_y} + \|X_t - \tilde{X}_t\|^2_{L^2_y(L^2_\omega)}}\Bigg).
\end{aligned}$$

We then obtain that

$$\sqrt{\|u-\tilde{u}\|^2_{L^2_y}+\|X_t-\tilde{X}_t\|^2_{L^2_y(L^2_\omega)}} \leqq C\int_0^t (|\dot{f}_0|+|\dot{f}_1|+e^{-s/2})\exp(-C(t-s))\,ds, \tag{60}$$

thus, using Lemma 3(ii) below,

$$\lim_{t\to\infty}\|u-\tilde{u}\|_{L^2_y}+\|X_t-\tilde{X}_t\|_{L^2_y(L^2_\omega)}=0. \tag{61}$$

For the stress, we have:

$$\begin{aligned}\int &|\mathbf{E}(X_t(y)Y_t(y))-\mathbf{E}(X_t^\infty Y_t^\infty)|\\ &\leqq \int \mathbf{E}|Y_t(y)(X_t(y)-X_t^\infty)|+\int \mathbf{E}|X_t^\infty(Y_t(y)-Y_t^\infty)|\\ &\leqq \|Y_t\|_{L^2_y(L^2_\omega)}\|X_t-X_t^\infty\|_{L^2_y(L^2_\omega)}+\|X_t^\infty\|_{L^2_y(L^2_\omega)}\|Y_t-Y_t^\infty\|_{L^2_y(L^2_\omega)},\end{aligned}$$

in which the right-hand side converges to zero by (52), (55) and (61). □

**Lemma 3.** *Let $k\in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ be a positive function, $\alpha>0$, and $h$ a function defined by:*

$$h(t)=\int_0^t \exp(-\alpha(t-s))k(s)\,ds,$$

*(i) if we assume that $k\in L^p(\mathbf{R}_+)$, with $1\leqq p<\infty$, then $h\in W^{1,p}(\mathbf{R}_+)$ and hence $\lim_{t\to\infty}h(t)=0$,*
*(ii) if we assume that $\lim_{t\to\infty}k(t)=0$, then $\lim_{t\to\infty}h(t)=0$,*
*(iii) if we suppose that $0\leqq k(t)\leqq Ce^{-\beta t}$, with $\beta>0$ and $\alpha\neq\beta$, then $h(t)\leqq \dfrac{C}{|\alpha-\beta|}e^{-\alpha\wedge\beta\, t}$.*

**Proof.** To prove assertion (i) we can check using the Hölder inequality and the Fubini Theorem, that $h\in L^p(\mathbf{R}_+)$ if $k\in L^p(\mathbf{R}_+)$. Therefore, $\dot{h}(t)=-\alpha h(t)+k(t)$ is in $L^p(\mathbf{R}_+)$. The fact that $h\in W^{1,p}(\mathbf{R}_+)$ then implies that $\lim_{t\to\infty}h(t)=0$. For assertion (ii), let us introduce $\varepsilon>0$. There exists $T>0$ such that $\forall s>T$, $0\leqq k(s)\leqq\varepsilon$. Dividing the integral defining $h$ into two parts (on $(0,T)$ and on $(T,t)$) and letting $t$ go to $\infty$ allows us to complete the proof. Assertion (iii) is obtained by a simple computation. □

**Remark 6** (Other assumptions on $f_0$ and $f_1$**).** Using Lemma 3(i) and the estimates (52)–(53)–(54)–(60), we can also show that the results of Lemma 2 hold under the following hypothesis, alternatively to (48)–(51): $(f_0(t)-a_0)\in W^{1,p}(\mathbf{R}_+)$ and $(f_1(t)-a_1)\in W^{1,p}(\mathbf{R}_+)$, for some $1\leqq p<\infty$.

In addition, by the same arguments and Lemma 3(iii), it can be shown that if $f_0, f_1\in W^{1,1}_{\mathrm{loc}}(\mathbf{R}_+)$ and $(f_0(t)-a_0)$, $(f_1(t)-a_1)$, $\dot{f}_0(t)$ and $\dot{f}_1(t)$ converge exponentially fast to 0, then the convergences stated in Lemma 2 are also exponential.

We are not able to extend the above arguments to the case of a nonlinear force (like the FENE force for example) or that of a geometry which is not a shear flow. In the next section, we present a more general approach, which requires us to manipulate the density $\psi(t,.)$ of the process $\boldsymbol{X}_t$.

## *3.2. Convergence to equilibrium by the entropy method for homogeneous Dirichlet boundary conditions*

In this section, we focus on the convergence to *equilibrium* ($\boldsymbol{u}_\infty = 0$, $\psi_\infty(\boldsymbol{X}) \propto \exp(-\Pi(\boldsymbol{X}))$) in a more general setting than in Section 3.1: we consider the long-time properties of the system (14)–(15)–(16)–(17) in a general geometry $\mathcal{D}$ and for any radially-symmetric $\alpha$-convex potential $\Pi$ (8). However, we restrict ourselves to homogeneous Dirichlet boundary conditions on the velocity: $\boldsymbol{u} = 0$ on $\partial\mathcal{D}$. The case of nonhomogeneous Dirichlet boundary conditions will be discussed later in Section 3.3.

**3.2.1. An energy estimate.** The energy estimate we have used to study the coupled PDE-SDE system in the case of a shear flow (see [19, 20]) can be established formally for any geometry. Multiplying (14) by $\boldsymbol{u}$, we have (using $\boldsymbol{u} = 0$ on $\partial\mathcal{D}$):

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{D}} |\boldsymbol{u}|^2 = -\int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2 - \int_{\mathcal{D}} \mathbf{E}(\boldsymbol{X}_t \otimes \nabla\Pi(\boldsymbol{X}_t)) : \nabla \boldsymbol{u}. \tag{62}$$

On the other hand, by Itô's calculus,

$$\begin{aligned}\frac{d}{dt}\mathbf{E}(\Pi(\boldsymbol{X}_t)) + \boldsymbol{u}.\nabla_{\boldsymbol{x}}\mathbf{E}(\Pi(\boldsymbol{X}_t)) &= \mathbf{E}(\nabla\Pi(\boldsymbol{X}_t).\nabla\boldsymbol{u}\, \boldsymbol{X}_t) - \frac{1}{2}\mathbf{E}(\|\nabla\Pi(\boldsymbol{X}_t)\|^2) \\ &\quad + \frac{1}{2}\mathbf{E}(\Delta\Pi(\boldsymbol{X}_t)).\end{aligned} \tag{63}$$

We thus obtain the following energy estimate:

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{D}} |\boldsymbol{u}|^2 + \int_{\mathcal{D}} \mathbf{E}(\Pi(\boldsymbol{X}_t))\right) + \int_{\mathcal{D}} |\nabla\boldsymbol{u}|^2 + \frac{1}{2}\int_{\mathcal{D}} \mathbf{E}(\|\nabla\Pi(\boldsymbol{X}_t)\|^2) \\ &\quad = \frac{1}{2}\int_{\mathcal{D}} \mathbf{E}(\Delta\Pi(\boldsymbol{X}_t)).\end{aligned} \tag{64}$$

Notice that we have assumed that $\forall \boldsymbol{x} \in \mathcal{D}$, $\forall t \geqq 0$, $\int_0^t \mathbf{E}(\|\nabla\Pi(\boldsymbol{X}_s)\|^2)\, ds < \infty$ to write that the local martingale $\int_0^t \nabla\Pi(\boldsymbol{X}_s) \,.\, d\boldsymbol{W}_s$ has zero mean, and that we have used the symmetry of the stress tensor to write: $\forall t \geqq 0$,

$$\mathbf{E}(\boldsymbol{X}_t \otimes \nabla\Pi(\boldsymbol{X}_t)) : \nabla\boldsymbol{u} = \mathbf{E}\left(\nabla\Pi(\boldsymbol{X}_t).\nabla\boldsymbol{u}\, \boldsymbol{X}_t\right). \tag{65}$$

The latter symmetry holds by construction for a radially symmetric potential $\Pi$. To remove the advective terms, we have also used the fact that div $\boldsymbol{u} = 0$ and $\boldsymbol{u} = 0$ on $\partial\mathcal{D}$.

In the case of a shear flow, both for Hookean dumbbells (see Section 3.2 in [19]) and for FENE dumbbells (see Lemma 4 in [20]), we have checked that these formal computations can be rigorously justified. In the case of a more general geometry and a general potential, they hold true as long as we assume that we manipulate a regular enough solution.

It is not clear how to use the *a priori* estimate (64) for studying the long-time behavior, since the term $\frac{1}{2}\int_{\mathcal{D}} \mathbf{E}(\Delta\Pi(\boldsymbol{X}_t))$ on the right-hand side is positive by the strict convexity of $\Pi$ (it is equal to $\frac{|\mathcal{D}|d}{2}$ for Hookean dumbbells, for example). This term "brings energy" into the system. In Section 3.1, we were able to get rid of this Itô term by using a coupling method; unfortunately, we are not able to generalize using the same method here. The aim of this section is to show that the use of a free energy (see (70) below) instead of the energy (64) also enables us to eliminate this Itô term and allows us to study the convergence of the system to equilibrium. However, contrary to the energy inequality (64) that can be stated either in the stochastic form or using the probability density function $\psi$, the free energy tool seems to be restricted to the analysis of the system in the Fokker-Planck form (14)–(15)–(16)–(17), since we introduce an entropy of the probability density function $\psi$. The latter cannot be simply expressed in terms of the stochastic process $\boldsymbol{X}_t$.

**3.2.2. Entropy and convergence to equilibrium.** Let us first introduce the kinetic energy:

$$E(t) = \frac{1}{2}\int_{\mathcal{D}} |\boldsymbol{u}|^2. \tag{66}$$

As we have already mentioned, we have (see (62)):

$$\frac{dE}{dt} = -\int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2 - \int_{\mathcal{D}}\int_{\mathbf{R}^d} (\boldsymbol{X} \otimes \nabla\Pi(\boldsymbol{X})) : \nabla \boldsymbol{u}\, \psi. \tag{67}$$

We now introduce the entropy of the system (in fact, the relative entropy with respect to $\psi_\infty$), namely:

$$\begin{aligned} H(t) &= \int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi(t,\boldsymbol{x},\boldsymbol{X}) \ln\left(\frac{\psi(t,\boldsymbol{x},\boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right), \\ &= \int_{\mathcal{D}}\int_{\mathbf{R}^d} \Pi(\boldsymbol{X})\psi(t,\boldsymbol{x},\boldsymbol{X}) + \int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi(t,\boldsymbol{x},\boldsymbol{X}) \ln(\psi(t,\boldsymbol{x},\boldsymbol{X})) + C, \end{aligned} \tag{68}$$

with

$$\psi_\infty(\boldsymbol{X}) = \frac{\exp(-\Pi(\boldsymbol{X}))}{\int_{\mathbf{R}^d} \exp(-\Pi(\boldsymbol{X}))}, \tag{69}$$

and $C = \ln(\int_{\mathbf{R}^d} \exp(-\Pi(\boldsymbol{X})))|\mathcal{D}|$.

After some computations (which will be detailed in Appendix A for a more general case), we obtain:

$$\frac{dH}{dt} = -\frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi \left|\nabla \ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 + \int_{\mathcal{D}}\int_{\mathbf{R}^d} (\boldsymbol{X} \otimes \nabla\Pi(\boldsymbol{X})) : \nabla \boldsymbol{u}\, \psi. \tag{70}$$

**Remark 7** (Boundary terms)**.** While deriving this estimate, some boundary terms appear due to integrations by parts (see (A.124) and (A.125) in Appendix A). To justify (70), we need these boundary terms to be zero. This is clear for Hookean dumbbells, but in the case of FENE dumbbells some assumptions are required on $\psi_0$ and on $\nabla \boldsymbol{u}$ (see (B.128) and (B.129)).

Therefore, if we consider the free energy of the system $F(t) = E(t) + H(t)$, we have:

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{D}} |\boldsymbol{u}|^2 + \int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi \ln\left(\frac{\psi}{\psi_\infty}\right)\right) + \int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2 + \frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi \left|\nabla \ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 = 0. \tag{71}$$

Comparison of (71) with (64) reflects the fact that the free energy, rather than the energy, is the quantity to consider.

On the basis of (71), we now proceed further. First, we are able to prove the uniqueness of the stationary state:

**Proposition 4.** *The unique stationary solution to the coupled problem with homogeneous Dirichlet boundary conditions on the velocity is:*

$$\boldsymbol{u} = \boldsymbol{u}_\infty = 0 \textit{ and } \psi = \psi_\infty \propto \exp(-\Pi).$$

**Proof.** It is easy to check that $(\boldsymbol{u}_\infty, \psi_\infty)$ is a stationary solution to the coupled problem for homogeneous Dirichlet boundary conditions. Equation (71) implies that any stationary state is equal to $(\boldsymbol{u}_\infty, \psi_\infty)$. □

In addition, we may prove the convergence to equilibrium:

**Proposition 5.** *Consider a solution to the coupled system (14)–(15)–(16)–(17) in the case of homogeneous Dirichlet boundary conditions on the velocity, and for some $\alpha$-convex potential $\Pi$. Then $\boldsymbol{u}$ converges exponentially fast in the $L^2_{\boldsymbol{x}}$ norm to $\boldsymbol{u}_\infty = 0$ and the entropy $\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi \ln\left(\frac{\psi}{\psi_\infty}\right)$, where $\psi_\infty \propto \exp(-\Pi)$, converges exponentially fast to 0. Therefore, $\psi$ converges exponentially fast in the $L^2_{\boldsymbol{x}}(L^1_{\boldsymbol{X}})$ norm to $\psi_\infty$.*

**Proof.** Inserting in (71):

(i) the Poincaré inequality[5]: $\forall \boldsymbol{u} \in H^1_0(\mathcal{D})$,

$$\int_{\mathcal{D}} |\boldsymbol{u}|^2 \leqq C_{\mathrm{PI}}(\mathcal{D}) \int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2, \tag{72}$$

(ii) and the Sobolev logarithmic inequality for $\psi_\infty$, which holds since the potential $\Pi$ is $\alpha$-convex (see [1, 2] and Section 2.1): $\forall \phi$ such that $\phi \geqq 0$ and $\int \phi = 1$,

$$\int_{\mathbf{R}^d} \phi \ln\left(\frac{\phi}{\psi_\infty}\right) \leqq C_{\mathrm{LSI}}(\psi_\infty) \int_{\mathbf{R}^d} \phi \left|\nabla \ln\left(\frac{\phi}{\psi_\infty}\right)\right|^2, \tag{73}$$

[5] To be consistent with the notation of Section 2.1, this inequality is a Poincaré inequality with respect to the density $\frac{1_{\mathcal{D}}}{|\mathcal{D}|}$ and $C_{\mathrm{PI}}(\mathcal{D}) = 2C_{\mathrm{PI}}\left(\frac{1_{\mathcal{D}}}{|\mathcal{D}|}\right)$.

we obtain $\dfrac{dF}{dt} \leqq -CF$, with $C = \min\left(\frac{2}{C_{\mathrm{PI}}(\mathcal{D})}, \frac{1}{2C_{\mathrm{LSI}}(\psi_\infty)}\right)$ and hence exponential convergence of $F$ to 0. This also implies the exponential convergence of $\|\boldsymbol{u}\|_{L^2_{\boldsymbol{x}}}$ to 0.

The Csiszar-Kullback inequality (41) allows us to obtain the exponential convergence of $\psi$ to $\psi_\infty$ in the $L^2_{\boldsymbol{x}}(L^1_{\boldsymbol{X}})$ norm. □

**Remark 8** (Other possible entropy functions)**.** In Section 2.1, we showed that the entropy method can be applied to study the long-time behavior of the Fokker-Planck equation with different forms of entropy (i.e., different functions $h$ with the notation of Section 2.1). The question might be asked whether it is also possible to study the coupled system (14)–(15)–(16)–(17) with another entropy other than the one we have chosen to consider, namely $h(x) = x\ln(x) - x + 1$.

In order to clarify this, we consider the case of a shear flow with homogeneous Dirichlet boundary conditions on the velocity (see Section 1.2), and perform the same computation as above in the case of a general entropy function $h$. We thus obtain:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{D}}|u|^2 + \int_{\mathcal{D}}\int_{\mathbf{R}^2} h\left(\frac{\psi}{\psi_\infty}\right)\psi_\infty\right) \\
&\quad + \int_{\mathcal{D}}\left|\frac{\partial u}{\partial y}\right|^2 + \frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^2}\left|\nabla\left(\frac{\psi}{\psi_\infty}\right)\right|^2 h''\left(\frac{\psi}{\psi_\infty}\right)\psi_\infty \\
&\quad = -\int_{\mathcal{D}}\int_{\mathbf{R}^2} Y\,\psi\,\frac{\partial u}{\partial y}\frac{\partial \Pi}{\partial X}\left(1 - \left(h'\left(\frac{\psi}{\psi_\infty}\right) - h\left(\frac{\psi}{\psi_\infty}\right)\frac{\psi_\infty}{\psi}\right)\right) \qquad (74)
\end{aligned}$$

instead of (71).

The right-hand side of (74) does not identically vanish nor has a sign, unless $h'(x) - h(x)/x = 1$, which yields with $h(1) = 0$, $h(x) = x\ln(x)$. Notice that this function $h$ defines the same entropy $H$ as for our choice $h(x) = x\ln(x) - x + 1$. Therefore, it seems that the "adapted" entropy function for the coupled system is indeed $h(x) = x\ln(x) - x + 1$. Such an argument, illustrating the coupling between the momentum equation on the velocity and the Fokker-Planck equation through the stress tensor, determines the entropy to consider.

One could argue that the expression (16) of the stress tensor has actually been derived from the entropy function $h(x) = x\ln(x) - x + 1$, by the principle of virtual works (see [11], Section 3.7.5 p. 75). But the Kramers expression (16) for the stress tensor may also be obtained independently from a specific choice of entropy, using simple physical arguments (see [11], Section 3.7.4 p. 72, or [27], Section 4.1.2 p. 158). In the latter case, the consideration of (74) somehow determines the entropy.

**3.2.3. Convergence of the stress tensor.** In this section, we would like to extend the results of Section 2.2 in the coupled framework. Additional difficulties appear since $\psi$ now depends on the space variable $\boldsymbol{x}$.

*3.2.3.1. The case of FENE dumbbells.* Let us start with the FENE model. We have only been able to show the "convergence" of the stress tensor in a very weak sense:

**Proposition 6.** *In the FENE model, if $b > 2$, for homogeneous Dirichlet boundary conditions on the velocity $\boldsymbol{u}$, we have*

$$\int_0^\infty \int_{\mathcal{D}} |\boldsymbol{\tau}(t, \boldsymbol{x}) - \boldsymbol{\tau}_\infty(\boldsymbol{x})| < \infty. \tag{75}$$

To prove Proposition 6, the following Lemma is required:

**Lemma 4.** *Consider the FENE potential $\Pi(\boldsymbol{X}) = -\frac{b}{2}\ln\left(1 - \frac{|\boldsymbol{X}|^2}{b}\right)$. Then, if $b > 2$, there exist $\eta < 1$ and $C_\eta > 0$ such that,*

$$\forall \boldsymbol{X} \in \mathcal{B}(0, \sqrt{b}),\ 0 \leqq \Delta\Pi(\boldsymbol{X}) \leqq \eta|\nabla\Pi(\boldsymbol{X})|^2 + C_\eta. \tag{76}$$

**Proof.** Simple computations give: $\forall \varepsilon > 0$,

$$\begin{aligned}
\Delta\Pi(\boldsymbol{X}) &= \frac{d + |\boldsymbol{X}|^2(2-d)/b}{(1 - |\boldsymbol{X}|^2/b)^2}, \\
&= \frac{d + |\boldsymbol{X}|^2(2-d)/b}{(1 - |\boldsymbol{X}|^2/b)^2} 1_{|X|^2 > b-\varepsilon} + \frac{d + |\boldsymbol{X}|^2(2-d)/b}{(1 - |\boldsymbol{X}|^2/b)^2} 1_{|X|^2 \leqq b-\varepsilon}, \\
&\leqq \frac{d + (2-d)(b-\varepsilon)/b}{b-\varepsilon} \frac{|\boldsymbol{X}|^2}{(1 - |\boldsymbol{X}|^2/b)^2} + \frac{d}{(\varepsilon/b)^2}, \\
&\leqq \frac{2 - \varepsilon(2-d)/b}{b-\varepsilon} |\nabla\Pi(\boldsymbol{X})|^2 + \frac{d}{(\varepsilon/b)^2}.
\end{aligned}$$

When $\varepsilon$ goes to 0, $\frac{2-\varepsilon(2-d)/b}{b-\varepsilon}$ goes to $2/b < 1$ and this completes the proof. □

**Remark 9.** Using Lemma 4 and a localization argument, it is possible to rigorously prove that (64) holds for FENE dumbbells, without assuming *a priori* that $\forall \boldsymbol{x} \in \mathcal{D}$, $\forall t \geqq 0$, $\int_0^t \mathbf{E}(\|\nabla\Pi(\boldsymbol{X}_s)\|^2)\, ds < \infty$.

We are now in position to prove Proposition 6.

**Proof.** In this proof we denote by $\mathcal{B}$ the ball centred at 0 with radius $\sqrt{b}$. Using Hölder inequalities, we have: $\forall t \geqq 0$,

$$\begin{aligned}
&\int_0^t \int_{\mathcal{D}} |\boldsymbol{\tau}(s, \boldsymbol{x}) - \boldsymbol{\tau}_\infty(\boldsymbol{x})| = \int_0^t \int_{\mathcal{D}} \left| \int_{\mathcal{B}} \frac{\boldsymbol{X} \otimes \boldsymbol{X}}{1 - |\boldsymbol{X}|^2/b} (\psi(s, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})) \right| \\
&\quad \leqq \int_0^t \int_{\mathcal{D}} \int_{\mathcal{B}} \left| \frac{\boldsymbol{X} \otimes \boldsymbol{X}}{1 - |\boldsymbol{X}|^2/b} \right| |\psi(s, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \\
&\quad \leqq \int_0^t \int_{\mathcal{D}} \left( \int_{\mathcal{B}} \left| \frac{\boldsymbol{X} \otimes \boldsymbol{X}}{1 - |\boldsymbol{X}|^2/b} \right|^2 |\psi(s, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^{1/2} \\
&\qquad \left( \int_{\mathcal{B}} |\psi(s, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^{1/2} \\
&\quad \leqq \int_0^t \left( \int_{\mathcal{D}} \left( \int_{\mathcal{B}} \left| \frac{\boldsymbol{X} \otimes \boldsymbol{X}}{1 - |\boldsymbol{X}|^2/b} \right|^2 |\psi(s, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^{2/3} \right)^{3/4} \\
&\qquad \left( \int_{\mathcal{D}} \left( \int_{\mathcal{B}} |\psi(s, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^2 \right)^{1/4}.
\end{aligned}$$

We know by Proposition 5 that there exists $\beta > 0$ such that $\forall s \geqq 0$,

$$\int_{\mathcal{D}} \left( \int_{\mathcal{B}} |\psi(s, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^2 \leqq C e^{-\beta s}.$$

For the first term in the right-hand side of the inequality, we have:

$$\begin{aligned}
&\int_{\mathcal{D}} \left( \int_{\mathcal{B}} \left| \frac{\boldsymbol{X} \otimes \boldsymbol{X}}{1 - |\boldsymbol{X}|^2/b} \right|^2 |\psi(s, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^{2/3} \\
&\quad \leqq |\mathcal{D}|^{1/3} \left( \int_{\mathcal{D}} \int_{\mathcal{B}} \left| \frac{\boldsymbol{X} \otimes \boldsymbol{X}}{1 - |\boldsymbol{X}|^2/b} \right|^2 |\psi(s, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})| \right)^{2/3} \\
&\quad \leqq |\mathcal{D}|^{1/3} \left( \int_{\mathcal{D}} \mathbf{E} \left| \frac{\boldsymbol{X}_s \otimes \boldsymbol{X}_s}{1 - |\boldsymbol{X}_s|^2/b} \right|^2 + |\mathcal{D}| \int_{\mathcal{B}} \left| \frac{\boldsymbol{X} \otimes \boldsymbol{X}}{1 - |\boldsymbol{X}|^2/b} \right|^2 \psi_\infty(\boldsymbol{X}) \right)^{2/3}.
\end{aligned}$$

If $b > 2$, it is easy to check that $\int_{\mathcal{B}} \left| \frac{\boldsymbol{X} \otimes \boldsymbol{X}}{1 - |\boldsymbol{X}|^2/b} \right|^2 \psi_\infty(\boldsymbol{X}) < \infty$. Therefore, we have:

$$\int_0^t \int_{\mathcal{D}} |\boldsymbol{\tau}(s, \boldsymbol{x}) - \boldsymbol{\tau}_\infty(\boldsymbol{x})| \leqq C \left( 1 + \int_0^t \left( \int_{\mathcal{D}} \mathbf{E} \left| \frac{\boldsymbol{X}_s \otimes \boldsymbol{X}_s}{1 - |\boldsymbol{X}_s|^2/b} \right|^2 \right)^{1/2} e^{-\beta s/4} \, ds \right),$$

where $C$ is a constant that does not depend on time.

Let us estimate the second term of the right-hand side:

$$\begin{aligned}
&\int_0^t \left( \int_{\mathcal{D}} \mathbf{E} \left| \frac{\boldsymbol{X}_s \otimes \boldsymbol{X}_s}{1 - |\boldsymbol{X}_s|^2/b} \right|^2 \right)^{1/2} e^{-\beta s/4} \, ds \\
&\quad \leqq \int_0^t \left( \int_{\mathcal{D}} \mathbf{E} \left( \frac{|\boldsymbol{X}_s|^4}{(1 - |\boldsymbol{X}_s|^2/b)^2} \right) \right)^{1/2} e^{-\beta s/4} \, ds \\
&\quad \leqq b^{1/2} \int_0^t \left( \int_{\mathcal{D}} \mathbf{E} \left( \frac{|\boldsymbol{X}_s|^2}{(1 - |\boldsymbol{X}_s|^2/b)^2} \right) e^{-\beta s/4} \right)^{1/2} e^{-\beta s/8} \, ds \\
&\quad \leqq b^{1/2} \left( \int_0^t \int_{\mathcal{D}} \mathbf{E} \left( \frac{|\boldsymbol{X}_s|^2}{(1 - |\boldsymbol{X}_s|^2/b)^2} \right) e^{-\beta s/4} \, ds \right)^{1/2} \left( \int_0^t e^{-\beta s/4} \, ds \right)^{1/2}.
\end{aligned}$$

Now, since (64) together with (76) in Lemma 4, implies: $\forall t \geqq 0$,

$$\int_0^t \int_{\mathcal{D}} \mathbf{E} \left( \frac{|\boldsymbol{X}_s|^2}{(1 - |\boldsymbol{X}_s|^2/b)^2} \right) < C(1 + t),$$

we easily derive by integration by parts that

$$\int_0^\infty \int_{\mathcal{D}} \mathbf{E} \left( \frac{|\boldsymbol{X}_s|^2}{(1 - |\boldsymbol{X}_s|^2/b)^2} \right) e^{-\beta s/4} \, ds < \infty,$$

which completes the proof. □

*3.2.3.2. The case of Hookean dumbbells.* In the case of Hookean dumbbells, where $\boldsymbol{X}_t$ is a Gaussian random variable, a more precise result than Proposition 6 may be proved.

**Proposition 7.** *In the Hookean model, with homogeneous Dirichlet boundary conditions on the velocity* $\boldsymbol{u}$, *there exist* $C, \beta > 0$, *such that* $\forall t \geqq 0$,

$$\int_{\mathcal{D}} |\boldsymbol{\tau}(t, \boldsymbol{x}) - \boldsymbol{\tau}_\infty(\boldsymbol{x})| \leqq C e^{-\beta t}. \tag{77}$$

**Proof.** Using the fact that $\boldsymbol{X}_t$ is Gaussian with zero mean (since for almost all $(t, \boldsymbol{x}, \boldsymbol{X})$, $\psi(t, \boldsymbol{x}, \boldsymbol{X}) = \psi(t, \boldsymbol{x}, -\boldsymbol{X})$), we know that $\psi(t, \boldsymbol{x}, \boldsymbol{X})$ is of the following form:

$$\psi(t, \boldsymbol{x}, \boldsymbol{X}) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Gamma_t)}} \exp\left(-\frac{\boldsymbol{X}.\Gamma_t^{-1}\boldsymbol{X}}{2}\right),$$

where $\Gamma_t = \mathbf{E}(\boldsymbol{X}_t \otimes \boldsymbol{X}_t) = \int_{\mathbf{R}^d} \boldsymbol{X} \otimes \boldsymbol{X}\, \psi(t, \boldsymbol{x}, \boldsymbol{X})\, d\boldsymbol{X}$ denotes the covariance matrix of $\boldsymbol{X}_t$, which depends on time and also on the space variable $\boldsymbol{x}$. The following explicit expression of the relative entropy can then be derived:

$$\int_{\mathbf{R}^d} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \ln\left(\frac{\psi(t, \boldsymbol{x}, \boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right) d\boldsymbol{X} = \frac{1}{2}\left(-\ln(\det(\Gamma_t)) - d + \operatorname{tr}(\Gamma_t)\right).$$

The covariance matrix $\Gamma_t$ is symmetric and nonnegative. Moreover, since for almost all $t \geqq 0$, $\int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \ln\left(\frac{\psi(t,\boldsymbol{x},\boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right) < \infty$, then for almost all $t \geqq 0$ and for almost all $\boldsymbol{x} \in \mathcal{D}$, $\Gamma_t$ is positive. Let us denote $\lambda_i(t, \boldsymbol{x}) > 0$ its eigenvalues $(1 \leqq i \leqq d)$, and $h$ the function defined on $\mathbf{R}_+^*$ by $h(x) = -\ln(x) - 1 + x$. Then it is easy to check that

$$\int_{\mathbf{R}^d} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \ln\left(\frac{\psi(t, \boldsymbol{x}, \boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right) d\boldsymbol{X} = \frac{1}{2}\sum_{i=1}^{d} h(\lambda_i(t, \boldsymbol{x})).$$

The function $h$ is such that $\exists c > 0$, $\forall x > 0$,

$$h(x) \geqq c \min((x-1)^2, |x-1|),$$

and thus, $\forall 1 \leqq i \leqq d$, $\int_{\mathcal{D}} \min((\lambda_i(t, \boldsymbol{x}) - 1)^2, |\lambda_i(t, \boldsymbol{x}) - 1|)$ converges exponentially fast to zero. By the Cauchy-Schwarz inequality, it is then easy to prove that $\forall 1 \leqq i \leqq d$, $\int_{\mathcal{D}} |\lambda_i(t, \boldsymbol{x}) - 1|$ converges exponentially fast to zero.

Now, using the fact that

$$\begin{aligned}\int_{\mathcal{D}} |\boldsymbol{\tau}(t, \boldsymbol{x}) - \boldsymbol{\tau}_\infty(\boldsymbol{x})| &= \int_{\mathcal{D}} \left| \int_{\mathbf{R}^d} \boldsymbol{X} \otimes \boldsymbol{X}\, (\psi(t, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})) \right| \\ &= \int_{\mathcal{D}} |\Gamma_t - \mathrm{Id}|, \\ &\leqq C \int_{\mathcal{D}} \sum_{i=1}^{d} |\lambda_i(t, \boldsymbol{x}) - 1|,\end{aligned}$$

we have proved the exponential convergence of $\int_{\mathcal{D}} \left| \int_{\mathbf{R}^d} \boldsymbol{X} \otimes \boldsymbol{X}(\psi(t, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{X})) \right|$ to zero. $\square$

### *3.3. Generalization of the entropy method to the case of non-homogeneous Dirichlet boundary conditions*

The entropy method we used in Section 3.2 is well suited for the convergence to equilibrium (i.e., $\boldsymbol{u}_\infty = 0$). In Section 3.2, the asymptotic regime was the equilibrium state, since we considered homogeneous Dirichlet boundary conditions on the velocity $\boldsymbol{u}$. When the asymptotic state is not the equilibrium state, the entropy method is more difficult to employ. In the present section, we show how it can be adapted to treat the case of non-homogeneous Dirichlet boundary conditions, and therefore non-zero stationary states. More precisely, we assume that $\boldsymbol{u} = g$ on $\partial\mathcal{D}$, where $g$ is a function defined on $\partial\mathcal{D}$ and *not dependent on time* [6].

Since we have non-zero boundary conditions on the velocity $\boldsymbol{u}$, we do not have, in general, an explicit expression of $\boldsymbol{u}_\infty$ and $\psi_\infty$. Therefore, unless otherwise stated, $(\boldsymbol{u}_\infty, \psi_\infty)$ is defined as a solution to the system (20)–(21)–(22)–(23). We will see that we are able to derive an estimate of the same kind as (71). However, two difficulties arise:

(i) in comparison to (71), some additional terms appear in the right-hand side of the free energy equality, and these need to be controlled,
(ii) *a priori* a logarithmic Sobolev inequality for $\psi_\infty$ is not known to hold.

We explain in this section how to circumvent these difficulties.

**3.3.1. Estimates for non-zero boundary conditions.** Let $(\boldsymbol{u}_\infty, \psi_\infty)$ be a solution to the system (20)–(21)–(22)–(23). We do not assume here an explicit expression for $(\boldsymbol{u}_\infty, \psi_\infty)$. In the following, we set $\overline{\boldsymbol{u}}(t, \boldsymbol{x}) = \boldsymbol{u}(t, \boldsymbol{x}) - \boldsymbol{u}_\infty(\boldsymbol{x})$ and $\overline{\psi}(t, \boldsymbol{x}, \boldsymbol{X}) = \psi(t, \boldsymbol{x}, \boldsymbol{X}) - \psi_\infty(\boldsymbol{x}, \boldsymbol{X})$. We also introduce the following quantities:

$$E(t) = \frac{1}{2}\int_{\mathcal{D}} |\overline{\boldsymbol{u}}|^2(t, \boldsymbol{x}), \tag{78}$$

$$H(t) = \int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \ln\left(\frac{\psi(t, \boldsymbol{x}, \boldsymbol{X})}{\psi_\infty(\boldsymbol{x}, \boldsymbol{X})}\right), \tag{79}$$

$$F(t) = E(t) + H(t). \tag{80}$$

By considering the derivative of $F$ with respect to time, we obtain after a lengthy computation that we demonstrate in Appendix A, the following free energy equality:

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\int_{\mathcal{D}} |\overline{\boldsymbol{u}}|^2 + \int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi \ln\left(\frac{\psi}{\psi_\infty}\right)\right) \\ &\quad + \int_{\mathcal{D}} |\nabla\overline{\boldsymbol{u}}|^2 + \frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi \left|\nabla_{\boldsymbol{X}} \ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 \\ &= -\int_{\mathcal{D}} \overline{\boldsymbol{u}}.\nabla\boldsymbol{u}_\infty\overline{\boldsymbol{u}} - \int_{\mathcal{D}}\int_{\mathbf{R}^d} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}(\ln\psi_\infty)\overline{\psi} \\ &\quad - \int_{\mathcal{D}}\int_{\mathbf{R}^d} (\nabla_{\boldsymbol{X}}(\ln\psi_\infty) + \nabla\Pi(\boldsymbol{X})).\nabla\overline{\boldsymbol{u}}\boldsymbol{X}\,\overline{\psi}. \end{aligned} \tag{81}$$

[6] In the following, we use the fact that $\overline{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_\infty$ is zero on $\partial\mathcal{D}$ to ensure that boundary terms vanish when performing some integrations by parts (see Appendix A).

We thus obtain three additional terms in the right-hand side compared to the case $\boldsymbol{u}_\infty = 0$ contained in (71). As already mentioned in Remark 7, the rigorous derivation of (81) for FENE dumbbells requires some technical assumptions on $\psi_0$, $\nabla \boldsymbol{u}$ and also $\psi_\infty$ (see Appendices A and B for details).

In order to check that the exponential convergence to the stationary state again holds in the present situation, we need to estimate these three additional terms and to prove that a logarithmic Sobolev inequality holds for $\psi_\infty$.

The first term,

$$-\int_{\mathcal{D}} \overline{\boldsymbol{u}}.\nabla \boldsymbol{u}_\infty \overline{\boldsymbol{u}}, \tag{82}$$

may be bounded by the left-hand side of (81) if $\nabla \boldsymbol{u}_\infty$ is small enough (in some norm to be made precise).

Similarly, the second term

$$-\int_{\mathcal{D}} \int_{\mathbf{R}^d} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}} (\ln \psi_\infty) \overline{\psi}, \tag{83}$$

may be bounded by the left-hand side of (81) if $\nabla_{\boldsymbol{x}} (\ln \psi_\infty)$ is small enough (in some norm to be made precise), using the Csiszar-Kullback inequality (41).

To estimate the third term,

$$\begin{aligned} &-\int_{\mathcal{D}} \int_{\mathbf{R}^d} (\nabla_{\boldsymbol{X}} (\ln \psi_\infty) + \nabla \Pi(\boldsymbol{X})) .\nabla \overline{\boldsymbol{u}} \boldsymbol{X}\, \overline{\psi} \\ &\quad = -\int_{\mathcal{D}} \int_{\mathbf{R}^d} \nabla_{\boldsymbol{X}} \left( \ln \left( \frac{\psi_\infty}{\exp(-\Pi)} \right) \right) .\nabla \overline{\boldsymbol{u}} \boldsymbol{X}\, \overline{\psi}, \end{aligned} \tag{84}$$

an estimate of $\left| \nabla_{\boldsymbol{X}} \left( \ln \left( \frac{\psi_\infty}{\exp(-\Pi)} \right) \right) \right| |\boldsymbol{X}|$ from above is needed. Notice that such an estimate is related to a logarithmic Sobolev inequality for $\psi_\infty$, since the latter holds with a constant $C$ which depends on osc $\left( \ln \left( \frac{\psi_\infty}{\exp(-\Pi)} \right) \right) < \infty$ (see Lemma 1). All this is discussed later in Section 3.3.3.

In the rest of the paper, we focus mainly on the case when $\boldsymbol{u}_\infty$ is a homogeneous flow (i.e., with a constant $\nabla \boldsymbol{u}_\infty$) and the potential is the FENE potential, since it is the only case for which we can control these three terms.

**3.3.2. Homogeneous stationary flows.** In this section, we show how the considered problem is simplified for homogeneous stationary flows. More precisely, we assume that the boundary conditions on $\boldsymbol{u}$ are such that a homogeneous flow $\boldsymbol{u}_\infty$ (i.e., $\boldsymbol{u}_\infty(\boldsymbol{x}) = \boldsymbol{\kappa x}$) satisfies the momentum equation (20)–(21) with div $(\boldsymbol{\tau}_\infty) = 0$. Notice that this imposes $\mathrm{tr}(\boldsymbol{\kappa}) = 0$ and $\boldsymbol{\kappa}^2$ be symmetric[7] since (20) then writes $\boldsymbol{\kappa}^2 \boldsymbol{x} = \nabla p_\infty$ thus $\mathrm{curl}(\boldsymbol{\kappa}^2 \boldsymbol{x}) = 0$, and, for any matrix $\boldsymbol{M}$, $\mathrm{curl}(\boldsymbol{M x}) = 0$ if, and only if, $\boldsymbol{M}$ is a symmetric matrix. Then, if we now define $\psi_\infty$ as a solution of (23) which does not depend on space, $(\boldsymbol{u}_\infty, \psi_\infty)$ is a solution to (20)–(21)–(22)–(23). Notice that the fact that $\psi_\infty$ does not depend on space implies that the term (83) vanishes.

[7] In dimension $d = 2$, the fact that $\boldsymbol{\kappa}$ is traceless actually implies that the off-diagonal terms of $\boldsymbol{\kappa}^2$ are zero, and *a fortiori* that $\boldsymbol{\kappa}^2$ is symmetric.

*3.3.2.1. The case when $\boldsymbol{\kappa}$ is antisymmetric.* The term (82) then vanishes while $\psi_\infty(\boldsymbol{X}) \propto \exp(-\Pi(\boldsymbol{X}))$ is a solution of (23). Then, using the fact that the term (84) is also zero, and that a logarithmic Sobolev inequality holds for $\psi_\infty$, we obtain the exponential convergence of $F$ to 0. Therefore, we have proved:

**Proposition 8** (The case when $\boldsymbol{\kappa}$ is antisymmetric)**.** *Consider a solution to the coupled system (14)–(15)–(16)–(17) in the case of non-homogeneous stationary Dirichlet boundary conditions on the velocity, and for an $\alpha$-convex potential* $\Pi$. *We assume that a homogeneous flow $\boldsymbol{u}_\infty(\boldsymbol{x}) = \boldsymbol{\kappa x}$ with $\boldsymbol{\kappa}$ antisymmetric satisfies the boundary conditions on $\boldsymbol{u}$. We set* $\psi_\infty(\boldsymbol{X}) \propto \exp(-\Pi(\boldsymbol{X}))$. *Then $(\boldsymbol{u}_\infty, \psi_\infty)$ is a solution of (20)–(21)–(22)–(23).*

*Moreover, $\boldsymbol{u}$ converges exponentially fast in the $L^2_{\boldsymbol{x}}$ norm to $\boldsymbol{u}_\infty$ and the entropy $\displaystyle\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi \ln\left(\frac{\psi}{\psi_\infty}\right)$ converges exponentially fast to 0. Therefore, $\psi$ converges exponentially fast in the $L^2_{\boldsymbol{x}}(L^1_{\boldsymbol{X}})$ norm to $\psi_\infty$.*

*3.3.2.2. The case when $\boldsymbol{\kappa}$ is symmetric.* If $\int_{\mathbf{R}^d} \exp(-\Pi(\boldsymbol{X}) + \boldsymbol{X}.\boldsymbol{\kappa X}) < \infty$, it is easy to check that $\psi_\infty(\boldsymbol{X}) \propto \exp(-\Pi(\boldsymbol{X}) + \boldsymbol{X}.\boldsymbol{\kappa X}))$ is a solution to (23). In this case, the term (84) becomes:

$$-2\int_{\mathcal{D}}\int_{\mathbf{R}^d} \boldsymbol{\kappa X}.\nabla\overline{\boldsymbol{u}}\boldsymbol{X}\,\overline{\psi}.$$

This will enable some specific computations for FENE dumbbells in Section 3.3.3. The fact that $\boldsymbol{\kappa}$ is symmetric is equivalent to $\mathrm{curl}(\boldsymbol{u}_\infty) = 0$ (such a flow is called a potential flow).

*3.3.2.3. The case of a shear flow.* Shear flow is a simple case where $\boldsymbol{u}_\infty(\boldsymbol{x}) = (u_\infty(y), 0)$ may be assumed to be homogeneous, with $\boldsymbol{\kappa}$ neither antisymmetric nor symmetric. Let us rewrite the free energy estimate in this special case. We choose the following boundary conditions: $u(t,0) = a_0$, $u(t,1) = a_1$ (which corresponds to $f_0(t) = a_0$ and $f_1(t) = a_1$ in Section 3.1). The expected stationary state is then

$$u_\infty(y) = a_0 + y(a_1 - a_0).$$

The function $\psi_\infty$ is defined as a solution to:

$$-\frac{\partial}{\partial X}\left((a_1 - a_0)Y\psi_\infty\right) - \mathrm{div}_{\,X,Y}\left(-\frac{1}{2}\psi_\infty\nabla\Pi\right) + \frac{1}{2}\Delta_{X,Y}\psi_\infty = 0. \quad (85)$$

Notice that the couple $((u_\infty, 0), \psi_\infty)$ is then a solution of the system (20)–(21)–(22)–(23).

In this particular geometry, since $\overline{\boldsymbol{u}}(t,\boldsymbol{x}) = (\overline{u}(t,y), 0)$, (81) reduces to:

$$\begin{aligned}\frac{dF}{dt} + \int_{\mathcal{D}}\left|\frac{\partial\overline{u}}{\partial y}\right|^2 + \frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^2}\psi\left|\nabla_X \ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 \\ = -\int_{\mathcal{D}}\int_{\mathbf{R}^2}\left(\frac{\partial(\ln\psi_\infty)}{\partial X} + \frac{\partial\Pi}{\partial X}(X,Y)\right)\frac{\partial\overline{u}}{\partial y}\,Y\overline{\psi}.\end{aligned} \quad (86)$$

Indeed, owing to the particular geometry, both advective terms (82) and (83) vanish.

*3.3.2.4. The case of Hookean dumbbells in a shear flow.* Let us elaborate further on the shear flow described above assuming in addition Hookean dumbbells. This case has been already studied using a coupling method, see Section 3.1, in a more general framework (since the Dirichlet boundary conditions were allowed to vary in time), but we want to show how to deal with it using an entropy method. Since the densities are Gaussian, an explicit solution to (85) is:

$$\psi_\infty(X,Y) \propto \exp\left(-\frac{(1+2(a_1-a_0)^2)Y^2 - 2(a_1-a_0)XY + X^2}{2(1+(a_1-a_0)^2)}\right).$$

Notice that, contrary to the case of a stationary homogeneous flow with a symmetric $\nabla \boldsymbol{u}_\infty$ (see above), $\psi_\infty$ does not satisfy the detailed balance (see Section 2.1).

The explicit expression of $\psi_\infty$ can be used to provide an alternative proof of convergence to that of Section 3.1, by modifying the definition of the free energy $F$. Indeed, considering the following free energy: $F(t) = E(t) + \lambda H(t)$, for some $\lambda > 0$, the term (84) becomes:

$$-\int_{\mathcal{D}}\int_{\mathbf{R}^d} \nabla_X \left(\ln\left(\frac{(\psi_\infty)^\lambda}{C\exp(-\Pi)}\right)\right).\nabla\overline{\boldsymbol{u}}\boldsymbol{X}\overline{\psi},$$

where $\lambda$ and $C$ are two constants which can be arbitrarily chosen. Then, by considering $\lambda = 1 + (a_1 - a_0)^2$, the term (84) reads (in our special setting of Hookean dumbbells in a shear flow):

$$-\int_{\mathcal{D}}\int_{\mathbf{R}^2} \frac{\partial}{\partial X}\left(\ln\left(\frac{(\psi_\infty)^{1+(a_1-a_0)^2}}{C\exp(-\Pi)}\right)\right)\frac{\partial \overline{u}}{\partial y} Y\overline{\psi} = -(a_1-a_0)\int_{\mathcal{D}}\frac{\partial \overline{u}}{\partial y}\int_{\mathbf{R}^2} Y^2\overline{\psi}.$$

The exponential convergence of $F$ to 0 can then be deduced from Lemma 3(iii), using the fact that $\int_{\mathbf{R}^2} Y^2\overline{\psi} = \mathbf{E}(Y_t^2) - \int_{\mathbf{R}^2} Y^2\psi_\infty$ does not depend on space and converges exponentially fast to 0, and the following Young inequality:

$$\left|-(a_1-a_0)\int_{\mathcal{D}}\frac{\partial \overline{u}}{\partial y}\int_{\mathbf{R}^2} Y^2\overline{\psi}\right| \le \varepsilon\int_{\mathcal{D}}\left|\frac{\partial \overline{u}}{\partial y}\right|^2 + \frac{(a_1-a_0)^2\,|\mathcal{D}|}{4\varepsilon}\left(\int_{\mathbf{R}^2} Y^2\overline{\psi}\right)^2.$$

**Remark 10** (Some remarks on the case of Hookean dumbbells)**.** We consider in this remark, the Hookean dumbbell case for a general homogeneous stationary flow. It is easy to derive by Itô's calculus that $\boldsymbol{\tau} = \mathbf{E}(\boldsymbol{X}_t \otimes \boldsymbol{X}_t)$ satisfies the following ODE:

$$\frac{d\boldsymbol{\tau}}{dt} + \boldsymbol{u}.\nabla\boldsymbol{\tau} = \nabla\boldsymbol{u}\boldsymbol{\tau} + \boldsymbol{\tau}(\nabla\boldsymbol{u})^T - \boldsymbol{\tau} + \mathrm{Id}. \tag{87}$$

By the characteristic method (see (33) and the beginning of Section 2), we can rewrite (87) in the following form:

$$\frac{d\boldsymbol{\tau}}{dt} = \boldsymbol{G}(t)\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{G}(t)^T - \boldsymbol{\tau} + \mathrm{Id}. \tag{88}$$

Since $\nabla\boldsymbol{u}_\infty = \boldsymbol{\kappa}$ does not depend on space, it is natural to look for a stationary stress $\boldsymbol{\tau}_\infty$ which does not depend on space, and which therefore is such that

$$\boldsymbol{\kappa}\boldsymbol{\tau}_\infty + \boldsymbol{\tau}_\infty\boldsymbol{\kappa}^T - \boldsymbol{\tau}_\infty + \mathrm{Id} = 0. \tag{89}$$

We can deduce from [30] that (89) admits a nonnegative solution if, and only if, the eigenvalues of $\boldsymbol{\kappa}$ have their real parts strictly smaller than $1/2$. In this case, the solution is unique and $\boldsymbol{\tau}_\infty$ admits the following explicit expression:

$$\boldsymbol{\tau}_\infty = \int_0^\infty e^{-t} \exp(\boldsymbol{\kappa}\, t) \exp(\boldsymbol{\kappa}^T\, t)\, dt. \tag{90}$$

Moreover, $\boldsymbol{\tau}_\infty$ is symmetric and invertible. If $[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T] = 0$, then (90) simplifies to $\boldsymbol{\tau}_\infty = (\mathrm{Id} - \boldsymbol{\kappa} - \boldsymbol{\kappa}^T)^{-1}$.

We now assume that the eigenvalues of $\boldsymbol{\kappa}$ have their real parts strictly smaller than $1/2$. If $\psi(0, \boldsymbol{x}, .)$ is Gaussian (with zero mean), so are $\psi(t, \boldsymbol{x}, .)$ and $\psi_\infty(.)$. Let us denote by $\Gamma_t(\boldsymbol{x})$ (resp. $\Gamma_\infty$) the covariance matrix of $\psi(t, \boldsymbol{x}, .)$ (resp. of $\psi_\infty(.)$). Notice that $\boldsymbol{\tau}(t, \boldsymbol{x}) = \Gamma_t(\boldsymbol{x})$ and $\boldsymbol{\tau}_\infty = \Gamma_\infty$. Then, (84) becomes equal to $-\int_{\mathcal{D}} \mathrm{tr} \nabla \overline{\boldsymbol{u}}^T\ (\Gamma_\infty - \mathrm{Id})\ (\Gamma_\infty^{-1}\Gamma_t - \mathrm{Id})$.

On the other hand,

$$\begin{aligned}\int_{\mathcal{D}} \int_{\mathrm{R}^d} &\psi(t, \boldsymbol{x}, \boldsymbol{X}) \ln\left(\frac{\psi(t, \boldsymbol{x}, \boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right) d\boldsymbol{X} \\ &= \int_{\mathcal{D}} \frac{1}{2} \Big(-\ln(\det(\Gamma_\infty^{-1}\Gamma_t)) - d + \mathrm{tr}(\Gamma_\infty^{-1}\Gamma_t)\Big),\end{aligned}$$

and

$$\begin{aligned}\int_{\mathcal{D}} \int_{\mathrm{R}^d} &\psi(t, \boldsymbol{x}, \boldsymbol{X}) \left|\nabla_{\boldsymbol{X}} \ln\left(\frac{\psi(t, \boldsymbol{x}, \boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right)\right|^2 d\boldsymbol{X} \\ &= \int_{\mathcal{D}} \mathrm{tr}((\Gamma_\infty^{-1}\Gamma_t - \mathrm{Id})^2 (\Gamma_\infty^{-1}\Gamma_t)^{-1}\Gamma_\infty^{-1}).\end{aligned}$$

If $\Gamma_\infty^{-1}\Gamma_t$ is too large (for example, if we start far from the stationary solution), it seems unclear to us how to control the term (84) by the left-hand side of (81). Indeed when $x \to \infty$, $\frac{1}{2}(-\ln(x) - 1 + x) \sim \frac{x}{2}$ and $\frac{(x-1)^2}{x} \sim x$, while the term (84) is quadratic.

**3.3.3. The case of FENE dumbbells.** In this section, we denote by $\mathcal{B}$ the ball centred at 0 with radius $\sqrt{b}$. In the FENE model, we know that $\psi(t, \boldsymbol{x}, \boldsymbol{X})$ and $\psi_\infty(\boldsymbol{x}, \boldsymbol{X})$ are zero if $|\boldsymbol{X}|^2 > b$. This will be useful in estimating the term (84) and $\mathrm{osc}\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right)$.

We restrict ourselves to the case of a stationary homogeneous flow (see Section 3.3.2): $\boldsymbol{u}_\infty(\boldsymbol{x}) = \boldsymbol{\kappa}\boldsymbol{x}$, where $\boldsymbol{\kappa}$ is a traceless matrix such that $\boldsymbol{\kappa}^2$ is symmetric.

We recall that if $\boldsymbol{\kappa}$ is antisymmetric, exponential convergence to a stationary solution holds (see Proposition 8). If $\boldsymbol{\kappa}$ is symmetric, then we have the following result:

**Theorem 1.** *In the case of a stationary potential homogeneous flow (which means that $\boldsymbol{u}_\infty(\boldsymbol{x}) = \boldsymbol{\kappa}\boldsymbol{x}$ with $\boldsymbol{\kappa}$ a symmetric matrix) in the FENE model, if*

$$C_{\mathrm{PI}}(\mathcal{D})|\boldsymbol{\kappa}| + 4b^2|\boldsymbol{\kappa}|^2 \exp(4b|\boldsymbol{\kappa}|) < 1, \tag{91}$$

*then* $\boldsymbol{u}$ *converges exponentially fast to* $\boldsymbol{u}_\infty$ *in the* $L^2_{\boldsymbol{x}}$ *norm and the entropy* $\int_{\mathcal{D}}\int_{\mathcal{B}} \psi \ln\left(\frac{\psi}{\psi_\infty}\right)$*, where* $\psi_\infty \propto \exp(-\Pi(\boldsymbol{X}) + \boldsymbol{X}.\boldsymbol{\kappa}\boldsymbol{X})$*, converges exponentially fast to 0. Therefore,* $\psi$ *converges exponentially fast in the* $L^2_{\boldsymbol{x}}(L^1_{\boldsymbol{X}})$ *norm to* $\psi_\infty$*.*

**Proof.** In this case, for any $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{dF}{dt} = &- \int_{\mathcal{D}} |\nabla \overline{\boldsymbol{u}}|^2 - \frac{1}{2}\int_{\mathcal{D}}\int_{\mathcal{B}} \psi \left|\nabla_{\boldsymbol{X}} \ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 - \int_{\mathcal{D}} \overline{\boldsymbol{u}}.\boldsymbol{\kappa}\overline{\boldsymbol{u}} \\
&- 2\int_{\mathcal{D}}\int_{\mathcal{B}} \boldsymbol{\kappa}\boldsymbol{X}.\nabla\overline{\boldsymbol{u}}\boldsymbol{X}\,\overline{\psi}, \\
\leqq &- (1-\varepsilon)\int_{\mathcal{D}} |\nabla \overline{\boldsymbol{u}}|^2 - \frac{1}{2C_{\mathrm{LSI}}(\psi_\infty)}\int_{\mathcal{D}}\int_{\mathcal{B}} \psi \ln\left(\frac{\psi}{\psi_\infty}\right) + |\boldsymbol{\kappa}|\int_{\mathcal{D}} |\overline{\boldsymbol{u}}|^2 \\
&+ \frac{b^2|\boldsymbol{\kappa}|^2}{\varepsilon}\int_{\mathcal{D}}\left(\int_{\mathcal{B}} |\overline{\psi}|\right)^2, \\
\leqq &\left(\frac{-(1-\varepsilon)}{C_{\mathrm{PI}}(\mathcal{D})} + |\boldsymbol{\kappa}|\right)\int_{\mathcal{D}} |\overline{\boldsymbol{u}}|^2 \\
&+ \left(\frac{4b^2|\boldsymbol{\kappa}|^2}{\varepsilon} - \frac{\exp(-4b|\boldsymbol{\kappa}|)}{2C_{\mathrm{LSI}}(\exp(-\Pi))}\right)\int_{\mathcal{D}}\int_{\mathcal{B}} \psi \ln\left(\frac{\psi}{\psi_\infty}\right),
\end{aligned} \tag{92}$$

where we have successively used the Csiszar-Kullback inequality (41), the Poincaré inequality (72), the logarithmic Sobolev inequality (73) and the fact that (see Lemma 1)

$$\begin{aligned}
C_{\mathrm{LSI}}(\psi_\infty) &\leqq C_{\mathrm{LSI}}(\exp(-\Pi)) \exp(2\,\mathrm{osc}(\boldsymbol{X}.\boldsymbol{\kappa}\boldsymbol{X} 1_{|\boldsymbol{X}|^2<b})), \\
&\leqq C_{\mathrm{LSI}}(\exp(-\Pi)) \exp(4b|\boldsymbol{\kappa}|),
\end{aligned}$$

where osc(.) is defined by (44). We know that $C_{\mathrm{LSI}}(\exp(-\Pi)) \leqq 1/2$ (see (43)) and therefore, exponential convergence of $F$ to 0 holds if

$$\exists \varepsilon > 0,\ \varepsilon < 1 - C_{\mathrm{PI}}(\mathcal{D})|\boldsymbol{\kappa}| \text{ and } \varepsilon > 4b^2|\boldsymbol{\kappa}|^2 \exp(4b|\boldsymbol{\kappa}|), \tag{93}$$

since in this case, the two terms between parentheses in (92) are negative. The fact that (93) is equivalent to (91) completes the proof. □

Let us now turn to the case of a general homogeneous flow. We decompose $\boldsymbol{\kappa}$ into its symmetric and antisymmetric parts:

$$\boldsymbol{\kappa}^s = \frac{\boldsymbol{\kappa} + \boldsymbol{\kappa}^T}{2}, \qquad \boldsymbol{\kappa}^a = \frac{\boldsymbol{\kappa} - \boldsymbol{\kappa}^T}{2}. \tag{94}$$

The function $\psi_\infty$ is now defined as a solution to (23).

To obtain an exponential convergence of $F$ to 0, a sufficient condition is to obtain some $L^\infty_{\boldsymbol{X}}$ estimate on $\nabla\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right)$ (see (84)). Indeed, this also yields an estimate on $\mathrm{osc}\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right)$ which gives a logarithmic Sobolev inequality on $\psi_\infty$ (see Lemma 1). This is the following result:

**Proposition 9.** *In the case of a stationary homogeneous flow (which means that* $\boldsymbol{u}_\infty(\boldsymbol{x}) = \boldsymbol{\kappa x}$*) for the FENE model, if*

$$M^2 b^2 \exp(4bM) + C_{\mathrm{PI}}(\mathcal{D})|\boldsymbol{\kappa}^s| < 1, \tag{95}$$

*where $M$ denotes the nondimensional number:*

$$M = \frac{1}{\sqrt{b}} \sup_{|\boldsymbol{X}|^2<b} \left| \nabla \left( \ln \left( \frac{\psi_\infty(\boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))} \right) \right) \right|,$$

*then $\boldsymbol{u}$ converges exponentially fast to $\boldsymbol{u}_\infty$ in the $L^2_{\boldsymbol{x}}$ norm and the entropy $\int_{\mathcal{D}} \int_{\mathcal{B}} \psi \ln \left( \frac{\psi}{\psi_\infty} \right)$ converges exponentially fast to 0. Therefore, $\psi$ converges exponentially fast in the $L^2_{\boldsymbol{x}}(L^1_{\boldsymbol{X}})$ norm to $\psi_\infty$.*

**Proof.** We have, for any $\varepsilon > 0$:

$$\begin{aligned}
\frac{dF}{dt} = &- \int_{\mathcal{D}} |\nabla \overline{\boldsymbol{u}}|^2 - \frac{1}{2} \int_{\mathcal{D}} \int_{\mathcal{B}} \psi \left| \nabla_{\boldsymbol{X}} \ln \left( \frac{\psi}{\psi_\infty} \right) \right|^2 - \int_{\mathcal{D}} \overline{\boldsymbol{u}}.\boldsymbol{\kappa}^s \overline{\boldsymbol{u}} \\
&- \int_{\mathcal{D}} \int_{\mathcal{B}} \nabla \left( \ln \left( \frac{\psi_\infty}{\exp(-\Pi)} \right) \right) . \nabla \overline{\boldsymbol{u}} \boldsymbol{X}\, \overline{\psi} \\
\leqq &- (1-\varepsilon) \int_{\mathcal{D}} |\nabla \overline{\boldsymbol{u}}|^2 - \frac{1}{2C_{\mathrm{LSI}}(\psi_\infty)} \int_{\mathcal{D}} \int_{\mathcal{B}} \psi \ln \left( \frac{\psi}{\psi_\infty} \right) + |\boldsymbol{\kappa}^s| \int_{\mathcal{D}} |\overline{\boldsymbol{u}}|^2 \\
&+ \frac{M^2 b^2}{4\varepsilon} \int_{\mathcal{D}} \left( \int_{\mathcal{B}} |\overline{\psi}| \right)^2 \\
\leqq &\left( \frac{-(1-\varepsilon)}{C_{\mathrm{PI}}(\mathcal{D})} + |\boldsymbol{\kappa}^s| \right) \int_{\mathcal{D}} |\overline{\boldsymbol{u}}|^2 \\
&+ \left( \frac{M^2 b^2}{\varepsilon} - \frac{\exp(-4bM)}{2C_{\mathrm{LSI}}(\exp(-\Pi))} \right) \int_{\mathcal{D}} \int_{\mathcal{B}} \psi \ln \left( \frac{\psi}{\psi_\infty} \right),
\end{aligned} \tag{96}$$

where we have used the fact that (see Lemma 1)

$$C_{\mathrm{LSI}}(\psi_\infty) \leqq C_{\mathrm{LSI}}(\exp(-\Pi)) \exp \left( 2 \operatorname{osc} \left( \ln \left( \frac{\psi_\infty}{\exp(-\Pi)} \right) \right) \right),$$

and

$$\begin{aligned}
\operatorname{osc} \left( \ln \left( \frac{\psi_\infty}{\exp(-\Pi)} \right) \right) &\leqq 2\sqrt{b} \sup_{|\boldsymbol{X}|^2<b} \left| \nabla \left( \ln \left( \frac{\psi_\infty}{\exp(-\Pi)} \right) \right) \right| \\
&\leqq 2bM.
\end{aligned}$$

We know that $C_{\mathrm{LSI}}(\exp(-\Pi)) \leqq 1/2$ (see (43)) and therefore, exponential convergence of $F$ to 0 holds if

$$\exists \varepsilon > 0,\ \varepsilon < 1 - C_{\mathrm{PI}}(\mathcal{D})|\boldsymbol{\kappa}^s| \text{ and } \varepsilon > M^2 b^2 \exp(4bM), \tag{97}$$

since in this case, the two terms between parentheses in (96) are negative. The fact that (97) is equivalent to (95) completes the proof. □

Actually, it is possible to obtain an $L^\infty_{\boldsymbol{X}}$ estimate on $\nabla\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right)$ for a stationary solution $\psi_\infty$ of the Fokker-Planck equation, by assuming that $|\boldsymbol{\kappa}^s| < 1/2$:

**Proposition 10** (Estimate on $\nabla\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right)$**).** *Let* $\Pi$ *be the FENE potential:* $\Pi(\boldsymbol{X}) = -\frac{b}{2}\ln\left(1 - \frac{|\boldsymbol{X}|^2}{b}\right)$ *and* $\boldsymbol{\kappa}$ *be a traceless matrix such that*

$$|\boldsymbol{\kappa}^s| < 1/2,$$

*where* $\boldsymbol{\kappa}^s$ *is defined by (94). There exists a unique nonnegative solution* $\psi_\infty \in \mathcal{C}^2(\mathcal{B}(0,\sqrt{b}))$ *of*

$$-\mathrm{div}\left(\left(\boldsymbol{\kappa}\boldsymbol{X} - \frac{1}{2}\nabla\Pi(\boldsymbol{X})\right)\psi_\infty(\boldsymbol{X})\right) + \frac{1}{2}\Delta\psi_\infty(\boldsymbol{X}) = 0 \text{ in } \mathcal{B}(0,\sqrt{b}), \quad (98)$$

*normalized by*

$$\int_{\mathcal{B}(0,\sqrt{b})}\psi_\infty = 1, \quad (99)$$

*and whose boundary behavior is characterized by:*

$$\inf_{\mathcal{B}(0,\sqrt{b})}\frac{\psi_\infty}{\exp(-\Pi)} > 0, \quad (100)$$

$$\sup_{\mathcal{B}(0,\sqrt{b})}\left|\nabla\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right| < \infty. \quad (101)$$

*Furthermore, it satisfies:* $\forall \boldsymbol{X} \in \mathcal{B}(0,\sqrt{b})$,

$$\left|\nabla\left(\ln\left(\frac{\psi_\infty(\boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))}\right)\right) - 2\boldsymbol{\kappa}^s\boldsymbol{X}\right| \leqq \frac{2\sqrt{b}\,|[\boldsymbol{\kappa},\boldsymbol{\kappa}^T]|}{1-2|\boldsymbol{\kappa}^s|}, \quad (102)$$

*where* $[.,.]$ *is the commutator bracket*[8]*:* $[\boldsymbol{\kappa},\boldsymbol{\kappa}^T] = \boldsymbol{\kappa}\boldsymbol{\kappa}^T - \boldsymbol{\kappa}^T\boldsymbol{\kappa}$.

**Proof.** Let us start with the uniqueness part. To this aim, we introduce three functions $\omega$, $\xi$ and $f$ defined by: $\forall \boldsymbol{X} \in \mathcal{B}(0,\sqrt{b})$,

$$\begin{aligned}\omega(\boldsymbol{X}) &= \exp(-\Pi(\boldsymbol{X})) = \left(1 - \frac{|\boldsymbol{X}|^2}{b}\right)^{b/2},\\ \xi(\boldsymbol{X}) &= \boldsymbol{\kappa}\boldsymbol{X},\\ f &= \frac{\psi_\infty}{\exp(-\Pi)}.\end{aligned}$$

[8] In dimension d $= 2$, $\mathrm{tr}(\boldsymbol{\kappa}) = 0$ and $[\boldsymbol{\kappa},\boldsymbol{\kappa}^T] = 0$ imply that $\boldsymbol{\kappa}$ is either symmetric or antisymmetric. This is false in dimension d $= 3$ $\left(\text{consider, } e.g., \boldsymbol{\kappa} = \begin{bmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\right)$.

Since, $\forall \boldsymbol{X} \in \mathcal{B}(0, \sqrt{b})$,

$$\omega(\boldsymbol{X}) \left( f(\boldsymbol{X})\xi(\boldsymbol{X}) - \frac{1}{2}\nabla f(\boldsymbol{X}) \right) = \boldsymbol{\kappa X}\psi_\infty(\boldsymbol{X}) - \frac{1}{2}\nabla\Pi(\boldsymbol{X})\psi_\infty(\boldsymbol{X}) - \frac{1}{2}\nabla\psi_\infty(\boldsymbol{X}),$$

$\xi$, $\omega$ and $f$ satisfy the assumptions of Lemma 5. In particular, (100) and (101) imply that $\int_{\mathcal{B}(0,\sqrt{b})} \psi_\infty > 0$ exists, so that (99) removes the undetermined multiplicative constant.

We now turn to the existence part. By standard existence results for stationary measures, there exists for any $0 < \rho < \sqrt{b}$, a function $\psi_\infty^\rho$ with

$$\psi_\infty^\rho \in \mathcal{C}^2(\overline{\mathcal{B}}(0, \rho)) \text{ and } \psi_\infty^\rho > 0 \text{ in } \overline{\mathcal{B}}(0, \rho)$$

solution of

$$\begin{cases} -\text{div } \left(\left(\boldsymbol{\kappa X} - \frac{1}{2}\nabla\Pi(\boldsymbol{X})\right)\psi_\infty^\rho(\boldsymbol{X})\right) + \frac{1}{2}\Delta\psi_\infty^\rho(\boldsymbol{X}) = 0 \text{ in } \mathcal{B}(0, \rho), \\ \left(-\left(\boldsymbol{\kappa X} + \frac{1}{2}\nabla\Pi(\boldsymbol{X})\right)\psi_\infty^\rho(\boldsymbol{X}) + \frac{1}{2}\nabla\psi_\infty^\rho(\boldsymbol{X})\right).\boldsymbol{n}(\boldsymbol{X}) = 0 \text{ on } \partial\mathcal{B}(0, \rho). \end{cases} \tag{103}$$

The fact that $\psi_\infty^\rho$ is positive follows from the gradient estimate in Lemma 6.

We normalize the solution by

$$\psi_\infty^\rho(\mathbf{0}) = 1. \tag{104}$$

According to Lemma 6, we have:

$$\sup_{\boldsymbol{X}\in\mathcal{B}(0,\rho)} \left| \nabla \left( \ln \left( \frac{\psi_\infty^\rho(\boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))} \right) \right) - \boldsymbol{\kappa}^s \boldsymbol{X} \right| \leqq \frac{2\sqrt{b}\,|[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]|}{1 - 2|\boldsymbol{\kappa}^s|}.$$

In particular, $\exists C$, $\forall 0 < \rho < \sqrt{b}$,

$$\sup_{\mathcal{B}(0,\rho)} \left| \nabla \left( \ln \left( \frac{\psi_\infty^\rho}{\exp(-\Pi)} \right) \right) \right| \leqq C. \tag{105}$$

Together with (104), (105) implies the existence of a constant $0 < C < \infty$ independent of $\rho$ such that, $\forall \boldsymbol{X} \in \mathcal{B}(0, \rho)$,

$$\frac{\psi_\infty^\rho(\boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))} \geqq \frac{1}{C} \text{ and } \left| \nabla \left( \frac{\psi_\infty^\rho(\boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))} \right) \right| \leqq C. \tag{106}$$

Let $\underline{\rho} \in (0, \sqrt{b})$. Using (106), and

$$\nabla\psi_\infty^\rho = \exp(-\Pi)\nabla \left( \frac{\psi_\infty^\rho}{\exp(-\Pi)} \right) - \nabla\Pi\, \psi_\infty^\rho,$$

it is obvious that for any $\rho \in (\underline{\rho}, \sqrt{b})$, $\|\nabla\psi_\infty^\rho\|_{L^\infty(\mathcal{B}(0,\underline{\rho}))}$ is bounded by a constant which does not depend on $\rho$. By (104), $\|\psi_\infty^\rho\|_{L^\infty(\mathcal{B}(0,\underline{\rho}))}$ is thus bounded by a constant which does not depend on $\rho \in (\underline{\rho}, \sqrt{b})$. Let $\alpha$ be a real in (0, 1). For any integer $k \geqq 0$, we denote $\mathcal{C}^{k,\alpha}(\overline{\mathcal{B}}(0, \underline{\rho}))$ the set of functions in $\mathcal{C}^k(\overline{\mathcal{B}}(0, \underline{\rho}))$, such that any derivative of order $k$ is $\alpha$-Hölder. An Interior-Schauder estimate for the strictly elliptic equation:

$$\frac{1}{2}\Delta\psi_\infty^\rho(\boldsymbol{X}) - \left(\boldsymbol{\kappa X} - \frac{1}{2}\nabla\Pi(\boldsymbol{X})\right).\nabla\psi_\infty^\rho(\boldsymbol{X})$$
$$-\text{div}\left(\boldsymbol{\kappa X} - \frac{1}{2}\nabla\Pi(\boldsymbol{X})\right)\psi_\infty^\rho(\boldsymbol{X}) = 0$$

yields that the functions $\{\psi_\infty^\rho\}_{\tilde{\rho}<\rho<\sqrt{b}}$ are uniformly bounded in the Hölder space $\mathcal{C}^{2,\alpha}(\overline{\mathcal{B}}(0, \underline{\rho}))$, where $\tilde{\rho} \in (\underline{\rho}, \sqrt{b})$ is fixed (see Theorem 6.2, p. 85, in [15]).

By the Arzela-Ascoli theorem, there exist an increasing sequence $\rho_n \to \sqrt{b}$ and a function $\psi_\infty \in \mathcal{C}^{2,\alpha}_{\text{loc}}(\mathcal{B}(0, \sqrt{b}))$ such that

$$\lim_{n\to\infty} \psi_\infty^{\rho_n} = \psi_\infty \text{ in } \mathcal{C}^2_{\text{loc}}(\mathcal{B}(0, \sqrt{b})).$$

In particular, by passing to the limit $\rho_n \to \sqrt{b}$ into (103), (105) and (106), we find that $\psi_\infty$ is a nonnegative solution of the partial differential equation (98) and satisfies the estimates (100), (101) and (102). In particular, (100) and (101) imply that $0 < \int_{\mathcal{B}(0,\sqrt{b})} \psi_\infty < \infty$, so that we may normalize $\psi_\infty$ to satisfy (99). □

**Lemma 5** (Uniqueness to (98))**.** *Assume that $\omega$ is a function such that*

$$\omega \in \mathcal{C}^1(\mathcal{B}(0, \sqrt{b})) \cap \mathcal{C}^0(\overline{\mathcal{B}}(0, \sqrt{b})), \tag{107}$$
$$\omega > 0 \textit{ in } \mathcal{B}(0, \sqrt{b}), \tag{108}$$
$$\omega = 0 \textit{ on } \partial\mathcal{B}(0, \sqrt{b}), \tag{109}$$

*and $\xi$ is a function such that*

$$\xi \in \mathcal{C}^1(\mathcal{B}(0, \sqrt{b})) \cap \mathcal{C}^0(\overline{\mathcal{B}}(0, \sqrt{b})). \tag{110}$$

*Therefore, up to multiplicative constants, there exists at most one function $f$ such that*

$$f \in \mathcal{C}^2(\mathcal{B}(0, \sqrt{b})), \quad \inf_{\mathcal{B}(0,\sqrt{b})} f > 0, \quad \sup_{\mathcal{B}(0,\sqrt{b})} |\nabla f| < \infty, \tag{111}$$

*with*

$$\text{div}\left(\omega\left(\xi f - \frac{1}{2}\nabla f\right)\right) = 0 \textit{ in } \mathcal{B}(0, \sqrt{b}). \tag{112}$$

**Proof.** For any functions $f, \tilde{f} \in \mathcal{C}^2(\mathcal{B}(0, \sqrt{b}))$ with $f, \tilde{f} > 0$ in $\mathcal{B}(0, \sqrt{b})$, we have:

$$\frac{1}{2}\tilde{f}\left|\nabla \ln\left(\frac{\tilde{f}}{f}\right)\right|^2 = \frac{1}{2}\left(\nabla \ln\left(\frac{\tilde{f}}{f}\right).\nabla\tilde{f} - \nabla\left(\frac{\tilde{f}}{f}\right).\nabla f\right),$$
$$= \nabla \ln\left(\frac{\tilde{f}}{f}\right).\left(\frac{1}{2}\nabla\tilde{f} - \xi\tilde{f}\right) - \nabla\left(\frac{\tilde{f}}{f}\right).\left(\frac{1}{2}\nabla f - \xi f\right). \tag{113}$$

Let us now consider two functions $f$ and $\tilde{f}$ which satisfy (111) and (112). By multiplying (113) by $\omega$, integrating over $\mathcal{B}(0,\rho)$ for a $0 < \rho < \sqrt{b}$ and then using some integrations by parts, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathcal{B}(0,\rho)} \tilde{f}\left|\nabla \ln\left(\frac{\tilde{f}}{f}\right)\right|^2 \omega \\
&\quad = \int_{\mathcal{B}(0,\rho)} \nabla \ln\left(\frac{\tilde{f}}{f}\right)\cdot\left(\frac{1}{2}\nabla\tilde{f} - \xi\tilde{f}\right)\omega - \int_{\mathcal{B}(0,\rho)} \nabla\left(\frac{\tilde{f}}{f}\right)\cdot\left(\frac{1}{2}\nabla f - \xi f\right)\omega, \\
&\quad = \int_{\partial\mathcal{B}(0,\rho)} \ln\left(\frac{\tilde{f}}{f}\right)\left(\frac{1}{2}\nabla\tilde{f} - \xi\tilde{f}\right).\boldsymbol{n}\,\omega - \int_{\partial\mathcal{B}(0,\rho)} \frac{\tilde{f}}{f}\left(\frac{1}{2}\nabla f + \xi f\right).\boldsymbol{n}\omega, \qquad (114)
\end{aligned}
$$

where $\boldsymbol{n}$ denotes the outward normal to $\mathcal{B}(0,\rho)$. According to (111), we have:

$$
\sup_{\mathcal{B}(0,\rho)}\left|\ln\left(\frac{\tilde{f}}{f}\right)\left(\frac{1}{2}\nabla\tilde{f} - \xi\tilde{f}\right)\right| < \infty \text{ and } \sup_{\mathcal{B}(0,\rho)}\left|\frac{\tilde{f}}{f}\left(\frac{1}{2}\nabla f + \xi f\right)\right| < \infty,
$$

so that, using (109), $\lim_{\rho\to\sqrt{b}}\int_{\partial\mathcal{B}(0,\rho)} \ln\left(\frac{\tilde{f}}{f}\right)\left(\frac{1}{2}\nabla\tilde{f} - \xi\tilde{f}\right).\boldsymbol{n}\,\omega = 0$ and $\lim_{\rho\to\sqrt{b}}\int_{\partial\mathcal{B}(0,\rho)} \frac{\tilde{f}}{f}\left(\frac{1}{2}\nabla f + \xi f\right).\boldsymbol{n}\,\omega = 0$. Passing to the limit $\rho \to \sqrt{b}$, (114) turns into $\int_{\mathcal{B}(0,\sqrt{b})} \tilde{f}\left|\nabla\ln\left(\frac{\tilde{f}}{f}\right)\right|^2 \omega = 0$. Due to (108) and (111), this yields as desired $\nabla\ln\left(\frac{\tilde{f}}{f}\right) = 0$ in $\mathcal{B}(0,\sqrt{b})$. □

**Lemma 6** (*A priori* estimate (102))**.** *Let* $0 < \rho < \sqrt{b}$ *and* $\psi_\infty \in \mathcal{C}^2(\overline{\mathcal{B}}(0,\rho))$ *be positive and satisfy*

$$
\begin{cases}
-\mathrm{div}\left(\left(\boldsymbol{\kappa}\boldsymbol{X} - \frac{1}{2}\nabla\Pi(\boldsymbol{X})\right)\psi_\infty(\boldsymbol{X})\right) + \frac{1}{2}\Delta\psi_\infty(\boldsymbol{X}) = 0 \;\; \text{in } \mathcal{B}(0,\rho), \\
\left(-\left(\boldsymbol{\kappa}\boldsymbol{X} + \frac{1}{2}\nabla\Pi(\boldsymbol{X})\right)\psi_\infty(\boldsymbol{X}) + \frac{1}{2}\nabla\psi_\infty(\boldsymbol{X})\right).\boldsymbol{n}(\boldsymbol{X}) = 0 \;\; \text{on } \partial\mathcal{B}(0,\rho).
\end{cases} \qquad (115)
$$

*If* $|\boldsymbol{\kappa}^s| < 1/2$, *then,* $\forall \boldsymbol{X} \in \mathcal{B}(0,\rho)$,

$$
\left|\nabla\left(\ln\left(\frac{\psi_\infty(\boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))}\right)\right) - 2\boldsymbol{\kappa}^s\boldsymbol{X}\right| \leqq \frac{2\sqrt{b}\,|[\boldsymbol{\kappa},\boldsymbol{\kappa}^T]|}{1-2|\boldsymbol{\kappa}^s|}. \qquad (116)
$$

**Proof.** Let $\boldsymbol{n}(\boldsymbol{X}) = \frac{\boldsymbol{X}}{\rho}$ denote the outward normal to $\mathcal{B}(0,\rho)$. Let $h(\boldsymbol{X}) = \ln\left(\frac{\psi_\infty(\boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))}\right) - \boldsymbol{X}.\boldsymbol{\kappa}^s\boldsymbol{X}$. Using the fact that $\ln(\psi_\infty(\boldsymbol{X})) = h(\boldsymbol{X}) - \Pi(\boldsymbol{X}) + \boldsymbol{X}.\boldsymbol{\kappa}^s\boldsymbol{X}$ and (A.121), it is easy to show that $h$ is such that

$$
\begin{aligned}
&2\boldsymbol{\kappa}\boldsymbol{X}.\nabla h - 2\boldsymbol{\kappa}\boldsymbol{X}.\nabla\Pi + 4\boldsymbol{\kappa}\boldsymbol{X}.\boldsymbol{\kappa}^s\boldsymbol{X} \\
&\quad = \nabla\Pi.\nabla h - |\nabla\Pi|^2 + 2\nabla\Pi.\boldsymbol{\kappa}^s\boldsymbol{X} + \Delta h + |\nabla h - \nabla\Pi + 2\boldsymbol{\kappa}^s\boldsymbol{X}|^2,
\end{aligned}
$$

so that

$$\Delta h + |\nabla h|^2 + (-\nabla \Pi + 2\boldsymbol{\kappa}.\boldsymbol{X}).\nabla h = 2(-\nabla \Pi + 2\boldsymbol{\kappa}^s \boldsymbol{X}).\boldsymbol{\kappa}^a \boldsymbol{X}.$$

Using the fact that $\nabla \Pi.\boldsymbol{\kappa}^a \boldsymbol{X} = 0$, which follows from the radial symmetry of $\Pi$, and that $\boldsymbol{\kappa}^s \boldsymbol{X}.\boldsymbol{\kappa}^a \boldsymbol{X} = -\frac{1}{4}\boldsymbol{X}.[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]\boldsymbol{X}$ we then obtain:

$$\Delta h + |\nabla h|^2 + (-\nabla \Pi + 2\boldsymbol{\kappa}^T \boldsymbol{X}).\nabla h = -\boldsymbol{X}.[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]\boldsymbol{X}. \tag{117}$$

Moreover, for any $\boldsymbol{X}$ on the boundary of $\mathcal{B}(0,\rho)$, $\nabla h.\boldsymbol{n} = \frac{1}{\psi_\infty}(\nabla \psi_\infty + (\nabla \Pi - 2\boldsymbol{\kappa}^s \boldsymbol{X})\psi_\infty)\,.\boldsymbol{n}$ so that, using $\boldsymbol{\kappa}^a \boldsymbol{X}.\boldsymbol{n} = 0$, the no-flux boundary condition on $\psi_\infty$, and the fact that $\psi_\infty$ is non-zero on the boundary, we have: $\forall \boldsymbol{X},\ |\boldsymbol{X}| = \rho$,

$$\nabla h(\boldsymbol{X}).\boldsymbol{n}(\boldsymbol{X}) = 0. \tag{118}$$

We now introduce $g = \frac{1}{2}|\nabla h|^2$. Notice that $\nabla g = D^2 h \nabla h$ and $\Delta g = \nabla(\Delta h).\nabla h + \mathrm{tr}(D^2 h D^2 h)$, so that, taking the gradient of (117) and multiplying by $\nabla h$, we obtain:

$$\begin{aligned}
&\Delta g - \mathrm{tr}(D^2 h D^2 h) + 2\nabla g.\nabla h + (-D^2 \Pi + 2\boldsymbol{\kappa}^T)\nabla h.\nabla h \\
&\quad + D^2 h(-\nabla \Pi + 2\boldsymbol{\kappa}^T \boldsymbol{X}).\nabla h \\
&\quad = -2[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]\boldsymbol{X}.\nabla h
\end{aligned}$$

and therefore

$$\begin{aligned}
\Delta g + (2\nabla h - \nabla \Pi + 2\boldsymbol{\kappa}^T \boldsymbol{X}).\nabla g = {} & D^2 \Pi \nabla h.\nabla h \\
& - 2\boldsymbol{\kappa}^s \nabla h.\nabla h + \mathrm{tr}(D^2 h D^2 h) \\
& - 2[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]\boldsymbol{X}.\nabla h.
\end{aligned}$$

We then obtain

$$\begin{aligned}
\Delta g + (2\nabla h - \nabla \Pi + 2\boldsymbol{\kappa}^T \boldsymbol{X}).\nabla g &\geqq (1 - 2|\boldsymbol{\kappa}^s|)|\nabla h|^2 - 2[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]\boldsymbol{X}.\nabla h \\
&\geqq 2(1 - 2|\boldsymbol{\kappa}^s|)\, g - 2\sqrt{2b}\,|[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]|\,\sqrt{g}.
\end{aligned}$$

The function $\phi : g \in \mathbf{R}_+ \mapsto 2(1 - 2|\boldsymbol{\kappa}^s|)\, g - 2\sqrt{2b}\,|[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]|\,\sqrt{g}$ is convex on $\mathbf{R}_+$ and such that $\phi(g^*) = 0$ and $\phi'(g^*) = (1 - 2|\boldsymbol{\kappa}^s|)$ where $g^* = \left(\frac{\sqrt{2b}\,|[\boldsymbol{\kappa},\boldsymbol{\kappa}^T]|}{1-2|\boldsymbol{\kappa}^s|}\right)^2$ so that

$$\phi(g) \geqq (1 - 2|\boldsymbol{\kappa}^s|)(g - g^*).$$

Therefore, we have:

$$\Delta(g - g^*) + (2\nabla h - \nabla \Pi + 2\boldsymbol{\kappa}^T \boldsymbol{X}).\nabla(g - g^*) \geqq (1 - 2|\boldsymbol{\kappa}^s|)(g - g^*). \tag{119}$$

Moreover, for any $\boldsymbol{X}$ on the boundary of $\mathcal{B}(0,\rho)$, we have

$$\nabla(g - g^*)(\boldsymbol{X}).\boldsymbol{n}(\boldsymbol{X}) = -\frac{|\nabla h(\boldsymbol{X})|^2}{|\boldsymbol{X}|} \leqq 0. \tag{120}$$

Indeed, let us introduce an orthonormal basis $(\boldsymbol{t}_1(\boldsymbol{X}), \boldsymbol{t}_2(\boldsymbol{X}), \boldsymbol{n}(\boldsymbol{X}))$ at point $\boldsymbol{X}$ such that $(\boldsymbol{t}_1(\boldsymbol{X}), \boldsymbol{t}_2(\boldsymbol{X}))$ is an orthonormal basis of the tangent plane to $\mathcal{B}(0, \rho)$ at point $\boldsymbol{X}$. Moreover, we can extend this basis in such a way that $(\boldsymbol{t}_1(\boldsymbol{X}), \boldsymbol{t}_2(\boldsymbol{X}), \boldsymbol{n}(\boldsymbol{X}))$ is constant in the $\boldsymbol{n}(\boldsymbol{X})$ direction. This can be easily done in spherical coordinates. Therefore, we have $\nabla \boldsymbol{t}_1(\boldsymbol{X})\boldsymbol{n}(\boldsymbol{X}) = \nabla \boldsymbol{t}_2(\boldsymbol{X})\boldsymbol{n}(\boldsymbol{X}) = \nabla \boldsymbol{n}(\boldsymbol{X})\boldsymbol{n}(\boldsymbol{X}) = 0$ and $\nabla \boldsymbol{n}(\boldsymbol{X})\boldsymbol{t}_i(\boldsymbol{X}) = \frac{1}{|\boldsymbol{X}|}\boldsymbol{t}_i(\boldsymbol{X})$.

Then, taking the tangential derivative of (118), we have ($i = 1, 2$): $\forall \boldsymbol{X},\ |\boldsymbol{X}| = \rho$,

$$
\begin{aligned}
0 &= \nabla(\nabla h(\boldsymbol{X}).\boldsymbol{n}(\boldsymbol{X})).\boldsymbol{t}_i(\boldsymbol{X}) \\
&= D^2 h(\boldsymbol{X})\boldsymbol{n}(\boldsymbol{X}).\boldsymbol{t}_i(\boldsymbol{X}) + \nabla h(\boldsymbol{X}).\nabla \boldsymbol{n}(\boldsymbol{X})\boldsymbol{t}_i(\boldsymbol{X}), \\
&= D^2 h(\boldsymbol{X})\boldsymbol{n}(\boldsymbol{X}).\boldsymbol{t}_i(\boldsymbol{X}) + \frac{1}{|\boldsymbol{X}|}\nabla h(\boldsymbol{X}).\boldsymbol{t}_i(\boldsymbol{X}).
\end{aligned}
$$

Therefore, we have (using again (118)):

$$
\begin{aligned}
\nabla g.\boldsymbol{n} &= D^2 h \nabla h.\boldsymbol{n}, \\
&= \nabla h.((D^2 h\boldsymbol{n}).\boldsymbol{t}_1\,\boldsymbol{t}_1 + (D^2 h\boldsymbol{n}).\boldsymbol{t}_2\,\boldsymbol{t}_2 + (D^2 h\boldsymbol{n}).\boldsymbol{n}\,\boldsymbol{n}), \\
&= -\frac{1}{|\boldsymbol{X}|}|\nabla h(\boldsymbol{X}).\boldsymbol{t}_1(\boldsymbol{X})|^2 - \frac{1}{|\boldsymbol{X}|}|\nabla h(\boldsymbol{X}).\boldsymbol{t}_2(\boldsymbol{X})|^2, \\
&= -\frac{|\nabla h(\boldsymbol{X})|^2}{|\boldsymbol{X}|},
\end{aligned}
$$

which yields (120).

Using the maximum principle on $(g - g^*)$ (see Equations (119)–(120)), if $|\boldsymbol{\kappa}^s| < \frac{1}{2}$, then $g - g^* \leqq 0$ (see [15] Theorem 3.5 p. 34), which is precisely the inequality (102). □

Combining the results of Propositions 9 and 10, we obtain:

**Theorem 2.** *In the case of a stationary homogeneous flow (which means that $\boldsymbol{u}_\infty(\boldsymbol{x}) = \boldsymbol{\kappa x}$) for the FENE model, if $|\boldsymbol{\kappa}^s| < \frac{1}{2}$, $\psi_\infty$ is the stationary solution built in Proposition 10 and*

$$
M^2 b^2 \exp(4bM) + C_{\mathrm{PI}}(\mathcal{D})|\boldsymbol{\kappa}^s| < 1,
$$

*where $M = 2|\boldsymbol{\kappa}^s| + \frac{2\,|[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]|}{1-2|\boldsymbol{\kappa}^s|}$, then $\boldsymbol{u}$ converges exponentially fast to $\boldsymbol{u}_\infty$ in the $L^2_{\boldsymbol{x}}$ norm and the entropy $\int_{\mathcal{D}}\int_{\mathcal{B}} \psi \ln\left(\frac{\psi}{\psi_\infty}\right)$ converges exponentially fast to 0. Therefore $\psi$ converges exponentially fast in the $L^2_{\boldsymbol{x}}(L^1_{\boldsymbol{X}})$ norm to $\psi_\infty$.*

**Proof.** If $|\boldsymbol{\kappa}^s| < \frac{1}{2}$, then, by Proposition 10,

$$
\sup_{|\boldsymbol{X}|^2 < b} \left|\nabla\left(\ln\left(\frac{\psi_\infty(\boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))}\right)\right)\right| \leqq M\sqrt{b},
$$

where $M = 2|\boldsymbol{\kappa}^s| + \frac{2\,|[\boldsymbol{\kappa}, \boldsymbol{\kappa}^T]|}{1-2|\boldsymbol{\kappa}^s|}$. Therefore, if $M^2 b^2 \exp(4bM) + C_{\mathrm{PI}}(\mathcal{D})|\boldsymbol{\kappa}^s| < 1$, Proposition 9 yields the results of the theorem. □

**Remark 11** (Uniqueness of the stationary state)**.** A corollary of Theorem 2 is the uniqueness of smooth stationary states which satisfies the hypotheses of Theorem 2. In particular, the uniqueness of a $\psi_\infty$ which satisfies the properties of Proposition 10 is also a corollary of Theorem 2.

**Remark 12** (General stationary flow)**.** In the case of a general stationary flow for the FENE model, we can prove exponential convergence provided the smallness of some quantities depending on the stationary state. For instance, if $\left\|\nabla\left(\ln\left(\frac{\psi_\infty}{\exp(-\Pi)}\right)\right)\right\|_{L^\infty_{\boldsymbol{x},X}}$, $\|\nabla_{\boldsymbol{x}} \ln(\psi_\infty)\|_{L^\infty_{\boldsymbol{x},X}}$ and $\|(\nabla \boldsymbol{u}_\infty)^s\|_{L^\infty_{\boldsymbol{x}}}$ are small enough, by a reasoning as in the proof of Proposition 9, we find that $\boldsymbol{u}$ converges exponentially fast to $\boldsymbol{u}_\infty$ in the $L^2_{\boldsymbol{x}}$ norm and the entropy $\int_{\mathcal{D}}\int_{\mathcal{B}} \psi \ln\left(\frac{\psi}{\psi_\infty}\right)$ converges exponentially fast to 0. However, we have not been able to establish these bounds on $(\boldsymbol{u}_\infty, \psi_\infty)$ and therefore, we will not elaborate further in that direction.

## 4. Conclusions and perspectives

In this paper, we have derived some *a priori* free energy and energy estimates to study the long-time behavior of a coupled system arising in the micro-macro modelling of polymeric fluids. We have shown that except in the special case of Hookean dumbbells in a shear flow, the quantity to be considered needs to contain an entropy term. Moreover, we have checked that the “adapted entropy function” suitable for the study of the coupled system is the “physical entropy” $h(x) = x\ln(x)$. With this entropy function, we have been able to prove formally exponential convergence to equilibrium in the case of homogeneous Dirichlet boundary conditions on the velocity $\boldsymbol{u}$. In the case of non-homogeneous Dirichlet boundary conditions on the velocity, the situation is more intricate, and we have only obtained exponential convergence to a stationary state for sufficiently small stationary solutions. We refer to Table 1 for a summary of the main results we have obtained.

It seems that the FENE dumbbell model behaves better in the long-time limit than the Hookean dumbbell model since:

(i) in the case of a stationary homogeneous potential flow, there always exists a stationary solution for FENE dumbbells, while this is false for Hookean dumbbells;
(ii) in the case of a homogeneous flow, it seems unclear to us how exponential convergence to the stationary state may hold for Hookean dumbbells, while we have been able to prove such convergence for FENE dumbbells, under adequate assumptions.

## Appendix A. Details of the computation of (81)

We give here the details of computations to obtain the free energy equality (81). We recall that we assume that $(\boldsymbol{u}, \psi)$ satisfies (14)–(15)–(16)–(17) while $(\boldsymbol{u}_\infty, \psi_\infty)$

**Table 1.** Summary of the main results obtained regarding the long-time convergence of the velocity $\boldsymbol{u}$, of the density $\psi$ and of the stress $\boldsymbol{\tau}$. A question mark "?" is used to represent an open problem. The notation "**DB**" means that the detailed balance (see (42)) holds for $\psi_\infty$

| | Hookean |
|---|---|
| Shear flow (Lemma 2, Remark 6) (Section 3.3.2) | exp. CV of $\boldsymbol{u}$ and $\boldsymbol{\tau}$ if the BC on $\boldsymbol{u}$ converges exp. fast to their stationary value. |
| $\boldsymbol{u} = 0$ on $\partial D$ **DB**, (Proposition 5, 6, 7) | exp. CV of $\boldsymbol{u}$ and $\psi \ln(\psi/\psi_\infty)$ exp. convergence of $\boldsymbol{\tau}$ |
| $\boldsymbol{u} \neq 0$ (and constant) on $\partial D$ | |
| $\boldsymbol{u}_\infty$ homog., $\nabla \boldsymbol{u}_\infty$ antisymm. (Proposition 8) | exp. CV of $\boldsymbol{u}$ and $\psi \ln(\psi/\psi_\infty)$ CV of $\boldsymbol{\tau}$ ? |
| $\boldsymbol{u}_\infty$ homog., $\nabla \boldsymbol{u}_\infty$ symm. (Theorem 1, Remark 10) **DB** | If $\nabla \boldsymbol{u}_\infty$ is too large, no stationary state for $\psi$ and $\boldsymbol{\tau}$. ? |
| $\boldsymbol{u}_\infty$ homogeneous (Theorem 2, Remark 10) | If $\nabla \boldsymbol{u}_\infty$ is too large, no stationary state for $\psi$ and $\boldsymbol{\tau}$. ? |
| $\boldsymbol{u}_\infty$ non-homogeneous | ? |

| | FENE |
|---|---|
| Shear flow (Lemma 2, Remark 6) (Section 3.3.2) | same results as $\boldsymbol{u}_\infty$ homogeneous |
| $\boldsymbol{u} = 0$ on $\partial D$ **DB**, (Proposition 5, 6, 7) | exp. CV of $\boldsymbol{u}$ and $\psi \ln(\psi/\psi_\infty)$ weak "convergence" of $\boldsymbol{\tau}$ |
| $\boldsymbol{u} \neq 0$ (and constant) on $\partial D$ | |
| $\boldsymbol{u}_\infty$ homog., $\nabla \boldsymbol{u}_\infty$ antisymm. (Proposition 8) | exp. CV of $\boldsymbol{u}$ and $\psi \ln(\psi/\psi_\infty)$ CV of $\boldsymbol{\tau}$ ? |
| $\boldsymbol{u}_\infty$ homog., $\nabla \boldsymbol{u}_\infty$ symm. (Theorem 1, Remark 10) **DB** | exp. CV of $\boldsymbol{u}$ and $\psi \ln(\psi/\psi_\infty)$ for small $\nabla \boldsymbol{u}_\infty$ CV of $\boldsymbol{\tau}$ ? |
| $\boldsymbol{u}_\infty$ homogeneous (Theorem 2, Remark 10) | exp. CV of $\boldsymbol{u}$ and $\psi \ln(\psi/\psi_\infty)$ if $(\nabla \boldsymbol{u}_\infty)^s$ and $M$ are small. CV of $\boldsymbol{\tau}$ ? |
| $\boldsymbol{u}_\infty$ non-homogeneous | ? (see Remark 12) |

satisfies (20)–(21)–(22)–(23). Moreover, we assume steady Dirichlet boundary conditions on $\boldsymbol{u}$ and $\boldsymbol{u}_\infty$: $\boldsymbol{u} = \boldsymbol{u}_\infty = g$ on $\partial\mathcal{D}$. The functions $E$, $H$ and $F$ are defined by (78), (79) and (80) respectively, while $\overline{\boldsymbol{u}}(t,\boldsymbol{x}) = \boldsymbol{u}(t,\boldsymbol{x}) - \boldsymbol{u}_\infty(\boldsymbol{x})$ and $\overline{\psi}(t,\boldsymbol{x},\boldsymbol{X}) = \psi(t,\boldsymbol{x},\boldsymbol{X}) - \psi_\infty(\boldsymbol{x},\boldsymbol{X})$.

Note first that the equation of $\psi_\infty$ (23) can be rewritten in the following manner:

$$\begin{aligned}
2\boldsymbol{u}_\infty(\boldsymbol{x}).\nabla_{\boldsymbol{x}}(\ln\psi_\infty(\boldsymbol{x},\boldsymbol{X})) &+ 2\nabla_{\boldsymbol{x}}\boldsymbol{u}_\infty(\boldsymbol{x})\boldsymbol{X}.\nabla_{\boldsymbol{X}}\ln(\psi_\infty(\boldsymbol{x},\boldsymbol{X})) \\
&= \Delta_{\boldsymbol{X}}\Pi(\boldsymbol{X}) + \nabla_{\boldsymbol{X}}\Pi(\boldsymbol{X}).\nabla_{\boldsymbol{X}}(\ln\psi_\infty(\boldsymbol{x},\boldsymbol{X})) + \Delta_{\boldsymbol{X}}(\ln\psi_\infty(\boldsymbol{x},\boldsymbol{X})) \\
&\quad + |\nabla_{\boldsymbol{X}}(\ln\psi_\infty(\boldsymbol{x},\boldsymbol{X}))|^2,
\end{aligned} \tag{A.121}$$

where we have used the fact that, for any smooth function $\phi$, $\Delta(\ln\phi) = \frac{\Delta\phi}{\phi} - |\nabla(\ln\phi)|^2$, and the fact that $\operatorname{div}_{\boldsymbol{X}}(\nabla_{\boldsymbol{x}}\boldsymbol{u}_\infty \boldsymbol{X}) = \operatorname{div}_{\boldsymbol{x}}\boldsymbol{u}_\infty = 0$.

We first make the computation neglecting the boundary terms (see below for their treatment). For the velocity, we easily obtain:

$$\begin{aligned}\frac{dE}{dt} = &-\int_{\mathcal{D}} |\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}|^2 - \int_{\mathcal{D}}\int_{\mathbf{R}^d} (\boldsymbol{X}\otimes\nabla_{\boldsymbol{X}}\Pi(\boldsymbol{X})) : \nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}\overline{\psi} \\ &-\int_{\mathcal{D}} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}\boldsymbol{u}_\infty\overline{\boldsymbol{u}}.\end{aligned} \tag{A.122}$$

Compared to the case of homogeneous Dirichlet boundary conditions on $\boldsymbol{u}$ (see (67)), the additional term comes from the nonlinearity of the advection term in the Navier-Stokes equations. Using (A.121), we have:

$$\begin{aligned}\frac{dH}{dt} = &-\frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^d} \frac{|\nabla_{\boldsymbol{X}}\psi|^2}{\psi} - \int_{\mathcal{D}}\int_{\mathbf{R}^d} \nabla_{\boldsymbol{X}}(\ln\psi_\infty).\nabla_{\boldsymbol{x}}\boldsymbol{u}\boldsymbol{X}\psi \\ &+\int_{\mathcal{D}}\int_{\mathbf{R}^d} \frac{1}{2}\left(\Delta_{\boldsymbol{X}}\Pi + \nabla_{\boldsymbol{X}}(\ln\psi_\infty).\nabla_{\boldsymbol{X}}\Pi - \Delta_{\boldsymbol{X}}(\ln\psi_\infty)\right)\psi \\ &-\int_{\mathcal{D}}\int_{\mathbf{R}^d} \boldsymbol{u}.\nabla_{\boldsymbol{x}}(\ln\psi_\infty)\psi, \\ = &-\frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^d} |\nabla_{\boldsymbol{X}}\ln\psi|^2\psi - \int_{\mathcal{D}}\int_{\mathbf{R}^d} \nabla_{\boldsymbol{X}}(\ln\psi_\infty)\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}\boldsymbol{X}\psi \\ &+\int_{\mathcal{D}}\int_{\mathbf{R}^d} \left(-\frac{1}{2}|\nabla_{\boldsymbol{X}}(\ln\psi_\infty)|^2 - \Delta_{\boldsymbol{X}}(\ln\psi_\infty)\right)\psi - \int_{\mathcal{D}}\int_{\mathbf{R}^d} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}(\ln\psi_\infty)\psi \\ = &-\frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi\left|\nabla_{\boldsymbol{X}}\ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 - \int_{\mathcal{D}}\int_{\mathbf{R}^d} \nabla_{\boldsymbol{X}}(\ln\psi_\infty)\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}\boldsymbol{X}\psi \\ &-\int_{\mathcal{D}}\int_{\mathbf{R}^d} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}(\ln\psi_\infty)\overline{\psi},\end{aligned} \tag{A.123}$$

where we have used the fact that $\int_{\mathcal{D}}\int_{\mathbf{R}^d} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}(\ln\psi_\infty)\psi_\infty = 0$ since $\operatorname{div}(\overline{\boldsymbol{u}}) = 0$ and $\overline{\boldsymbol{u}} = 0$ on $\partial\mathcal{D}$. Adding up (A.122) and (A.123) and using (65), we obtain:

$$\begin{aligned}\frac{dF}{dt} = &-\int_{\mathcal{D}} |\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}|^2 - \frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi\left|\nabla_{\boldsymbol{X}}\ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 - \int_{\mathbf{R}^d} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}\boldsymbol{u}_\infty\overline{\boldsymbol{u}} \\ &-\int_{\mathcal{D}}\int_{\mathbf{R}^d} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}(\ln\psi_\infty)\overline{\psi} - \int_{\mathcal{D}}\int_{\mathbf{R}^d} \nabla_{\boldsymbol{X}}(\ln\psi_\infty).\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}\boldsymbol{X}\psi \\ &-\int_{\mathcal{D}}\int_{\mathbf{R}^d} \nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}(t,\boldsymbol{x})\boldsymbol{X}.\nabla_{\boldsymbol{X}}\Pi(\boldsymbol{X})\overline{\psi} \\ = &-\int_{\mathcal{D}} |\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}|^2 - \frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi\left|\nabla_{\boldsymbol{X}}\ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 - \int_{\mathcal{D}} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}\boldsymbol{u}_\infty\overline{\boldsymbol{u}} \\ &-\int_{\mathcal{D}}\int_{\mathbf{R}^d} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}(\ln\psi_\infty)\overline{\psi} - \int_{\mathcal{D}}\int_{\mathbf{R}^d} \left(\nabla_{\boldsymbol{X}}(\ln\psi_\infty) + \nabla_{\boldsymbol{X}}\Pi(\boldsymbol{X})\right).\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}\boldsymbol{X}\psi \\ &+\int_{\mathcal{D}}\int_{\mathbf{R}^d} \nabla_{\boldsymbol{X}}\Pi(\boldsymbol{X}).\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}(t,\boldsymbol{x})\boldsymbol{X}\psi_\infty\end{aligned}$$

$$= -\int_{\mathcal{D}} |\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}|^2 - \frac{1}{2}\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi \left|\nabla_{\boldsymbol{X}} \ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 - \int_{\mathcal{D}} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}\boldsymbol{u}_\infty\overline{\boldsymbol{u}}$$
$$- \int_{\mathcal{D}}\int_{\mathbf{R}^d} \overline{\boldsymbol{u}}.\nabla_{\boldsymbol{x}}(\ln\psi_\infty)\overline{\psi} - \int_{\mathcal{D}}\int_{\mathbf{R}^d} \left(\nabla_{\boldsymbol{X}}(\ln\psi_\infty) + \nabla_{\boldsymbol{X}}\Pi(\boldsymbol{X})\right).\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}\boldsymbol{X}\,\overline{\psi},$$

where we have used the fact that

$$\int_{\mathcal{D}}\int_{\mathbf{R}^d} \nabla_{\boldsymbol{X}}(\ln\psi_\infty).\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}(t,\boldsymbol{x})\boldsymbol{X}\psi_\infty = \int_{\mathcal{D}}\int_{\mathbf{R}^d} \nabla_{\boldsymbol{X}}\psi_\infty.\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}(t,\boldsymbol{x})\boldsymbol{X}, \\ = 0,$$

since $\operatorname{div}_{\boldsymbol{X}}(\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}(t,\boldsymbol{x})\boldsymbol{X}) = \operatorname{div}_{\boldsymbol{x}}(\overline{\boldsymbol{u}}(t,\boldsymbol{x})) = 0$. This proves the free energy equality (81).

If we keep track of all the boundary terms, we actually obtain the following additional terms in the right-hand side of (81):

(i) boundary terms on $\partial\mathcal{D}$ (with outward normal $\boldsymbol{\nu}$):

$$B_1 = \int_{\partial\mathcal{D}} \overline{\boldsymbol{u}}.(\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}\,\boldsymbol{\nu}) - \int_{\partial\mathcal{D}} \overline{p}\,\overline{\boldsymbol{u}}.\boldsymbol{\nu} + \int_{\partial\mathcal{D}} \overline{\boldsymbol{u}}.(\overline{\boldsymbol{\tau}}\,\boldsymbol{\nu}) - \int_{\partial\mathcal{D}} \overline{\boldsymbol{u}}.\boldsymbol{\nu}$$
$$- \frac{1}{2}\int_{\partial\mathcal{D}} |\overline{\boldsymbol{u}}|^2\boldsymbol{u}.\boldsymbol{\nu} - \int_{\partial\mathcal{D}} \boldsymbol{u}.\boldsymbol{\nu}\int_{\mathbf{R}^d} \psi\ln\left(\frac{\psi}{\psi_\infty}\right), \tag{A.124}$$

(ii) boundary terms on $\partial\mathcal{B}(0,\rho)$ (with outward normal $\boldsymbol{n}$), with $\rho = \sqrt{b}$ in the case of a potential $\Pi(\boldsymbol{X})$ which is finite if, and only if, $|\boldsymbol{X}|^2 < b$ (in the FENE case for example), and $\rho \to \infty$ in the case of a potential $\Pi$ that is finite on $\mathbf{R}^d$:

$$B_2 = \int_{\mathcal{D}}\int_{|\boldsymbol{X}|=\rho} \left(\left(-\nabla_{\boldsymbol{x}}\boldsymbol{u}\boldsymbol{X} + \frac{1}{2}\nabla_{\boldsymbol{X}}\Pi\right)\psi + \frac{1}{2}\nabla_{\boldsymbol{X}}\psi\right).\boldsymbol{n}\ln\left(\frac{\psi}{\psi_\infty}\right)$$
$$+ \frac{1}{2}\int_{\mathcal{D}}\int_{|\boldsymbol{X}|=\rho} \nabla_{\boldsymbol{X}}\psi.\boldsymbol{n} - \frac{1}{2}\int_{\mathcal{D}}\int_{|\boldsymbol{X}|=\rho} \psi\nabla_{\boldsymbol{X}}\ln(\psi_\infty).\boldsymbol{n}$$
$$- \int_{\mathcal{D}} \nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}} : \int_{|\boldsymbol{X}|=\rho} \boldsymbol{n}\otimes\boldsymbol{X}\psi_\infty. \tag{A.125}$$

We see that that $B_1 = 0$, since $\overline{\boldsymbol{u}}$ and $\boldsymbol{u}.\boldsymbol{\nu}\int_{\mathbf{R}^d}\psi\ln\left(\frac{\psi}{\psi_\infty}\right)$ are equal to zero on $\partial\mathcal{D}$ (see 24). More rigorously, we only require $\overline{\boldsymbol{u}}$, $\psi$ and $\psi_\infty$ be regular enough in the $\boldsymbol{x}$ variable so that $\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}\,\boldsymbol{\nu} < \infty$, $\overline{p} < \infty$ and $\overline{\boldsymbol{\tau}}\,\boldsymbol{\nu} < \infty$ on $\partial\mathcal{D}$.

With regards to $B_2$, let us focus on the two prototypical potentials we have considered so far: Hookean and FENE dumbbells. For Hookean dumbbells, using the fact that $\psi$ and $\psi_\infty$ are Gaussian, it is easy to check that $\lim_{\rho\to\infty} B_2 = 0$. For FENE dumbbells, we need the fact that $\psi$ and $\psi_\infty$ are regular enough in the $\boldsymbol{X}$ variable to define their value and their normal derivative on the boundary $\partial\mathcal{B}(0,\sqrt{b})$. Moreover, if we assume that they decay on the boundary as $\exp(-\Pi)$ in the following sense: $\forall\boldsymbol{x}\in\mathcal{D}$, $\exists c, C > 0$, $\forall\boldsymbol{X}$, $|\boldsymbol{X}|^2 = b$,

$$c \leqq \frac{\psi_\infty(\boldsymbol{x},\boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))} \leqq C, \tag{A.126}$$

and $\forall t \geqq 0$, $\forall \boldsymbol{x} \in \mathcal{D}$, $\exists c, C > 0$, $\forall \boldsymbol{X}$, $|\boldsymbol{X}|^2 = b$,

$$c \leqq \frac{\psi(t, \boldsymbol{x}, \boldsymbol{X})}{\exp(-\Pi(\boldsymbol{X}))} \leqq C, \tag{A.127}$$

then, as the no-flux boundary condition on $\psi$ yields the first term in $B_2$ as zero, and as $\psi_\infty$ is zero on the boundary, this gives the last term in $B_2$ as zero also. Using the fact that $|\nabla \Pi| \exp(-\Pi)$ is zero on the boundary if $b > 2$, we can deduce from the no-flux boundary condition on $\psi$ (resp. on $\psi_\infty$) that $\nabla_{\boldsymbol{X}} \psi.\boldsymbol{n}$ (resp. $\nabla_{\boldsymbol{X}} \psi_\infty.\boldsymbol{n}$) is also zero on the boundary. Therefore, the second term in $B_2$ is also zero. For the third term, we write that $\int_{\mathcal{D}} \int_{|\boldsymbol{X}|^2=b} \psi \nabla_{\boldsymbol{X}} \ln(\psi_\infty).\boldsymbol{n} = \int_{\mathcal{D}} \int_{|\boldsymbol{X}|^2=b} \frac{\psi}{\psi_\infty} \nabla_{\boldsymbol{X}} \psi_\infty.\boldsymbol{n}$ which is also zero on the boundary.

We need now to justify that (A.126) and (A.127) hold. Concerning (A.126), we know that we can choose a $\psi_\infty$ which satisfies (A.126) in the following cases:

(i) homogeneous Dirichlet boundary conditions on $\boldsymbol{u}$,
(ii) stationary homogeneous flow, with $\nabla \boldsymbol{u}_\infty$ symmetric or antisymmetric,
(iii) stationary homogeneous flow and $(\boldsymbol{u}_\infty, \psi_\infty)$ satisfy the assumptions of Proposition 10.

Therefore, (A.126) holds in all the long-time convergence results we have stated in the propositions or theorems.

With regards to (A.127), we present in Appendix B a framework for the Fokker-Planck equation in which (A.127) holds.

## Appendix B. A variational formulation for the Fokker-Planck equation and a proof of (A.127)

We consider here the case of FENE dumbbells, so that $\exp(-\Pi) = (1-|X|^2/b)^{b/2}$ and we denote by $\mathcal{B}$ the ball centered at 0 with radius $\sqrt{b}$.

In this appendix, we would like to show that the boundary terms $B_2$ do indeed vanish by explaining why (A.127) holds if $\psi$ is a solution to (17) in a natural sense and under the following hypotheses:

$$\exists 0 < c \leqq C,\ c \leqq \frac{\psi_0}{\exp(-\Pi)} \leqq C, \tag{B.128}$$

$$\forall T > 0,\ \nabla \boldsymbol{u} \in L^\infty_{(0,T)}(L^\infty_{\mathcal{D}}). \tag{B.129}$$

Using (B.129), we can treat the advection term in (17) by the characteristic method (see (33) and the beginning of Section 2), so that we neglect this term in what follows. A natural framework (see [4, 9, 2]) to look for a solution $\psi$ to (17) is to consider $f = \frac{\psi}{\exp(-\Pi)}$ and to consider the variational formulation: find $f \in L^2_t(H^1_{\exp(-\Pi)})$ such that, $\forall g \in H^1_{\exp(-\Pi)}$,

$$\frac{d}{dt} \int_{\mathcal{B}} fg \exp(-\Pi) + \frac{1}{2} \int_{\mathcal{B}} \nabla f \, \nabla g \exp(-\Pi) = \int_{\mathcal{B}} \boldsymbol{G}(t)\boldsymbol{X}.\nabla g \, f \exp(-\Pi), \tag{B.130}$$

where $\boldsymbol{G}$ is the $L^\infty_{t,\mathrm{loc}}$ function defined by (34). We define the weighted Sobolev space $H^m_{\exp(-\Pi)}$ by: for $m \geqq 1$,

$$H^m_{\exp(-\Pi)} = \left\{ f \in \mathcal{D}'(\Omega),\ \sum_{|\alpha| \leqq m} \int_{\mathcal{B}} |D^\alpha f|^2 \exp(-\Pi) < \infty \right\},$$

with its norm $||f||_{H^m_{\exp(-\Pi)}} = \left( \sum_{|\alpha|\leqq m} \int_{\mathcal{B}} |D^\alpha f|^2 \exp(-\Pi) \right)^{1/2}$, and for $m = 0$,

$$H^0_{\exp(-\Pi)} = \left\{ f \in \mathcal{D}'(\Omega),\ \int_{\mathcal{B}} |f|^2 \exp(-\Pi) < \infty \right\},$$

with its norm $||f||_{H^0_{\exp(-\Pi)}} = \left( \int_{\mathcal{B}} |f|^2 \exp(-\Pi) \right)^{1/2}$. These spaces are Hilbert spaces, such that $\mathcal{C}^\infty(\overline{\mathcal{B}})$ is a dense subset. Notice that since $\exp(-\Pi) > 0$ in $\mathcal{B}$, $H^m_{\exp(-\Pi),\mathrm{loc}} = H^m_{\mathrm{loc}}$. For further properties of these spaces, we refer to [32], Chapter 3.

Using either a Galerkin method, or the Hille-Yosida theorem, we can prove that there exists a unique solution $f$ to (B.130) provided that $f_0 \in H^0_{\exp(-\Pi)}$. Let us now turn to the proof of (A.127).

**Proposition 11** (Proof of (A.127))**.** *Let us suppose that $f_0$ and $\boldsymbol{u}$ satisfy (B.128) and (B.129), respectively. Let $f$ be the solution to (B.130). Then, $\forall T > 0, \exists c, C, \mu > 0$, $\forall t \in [0, T]$, $\forall \boldsymbol{X} \in \mathcal{B}$,*

$$c \exp(-\mu t) \leqq f(t, \boldsymbol{X}) \leqq C \exp(\mu t). \tag{B.131}$$

**Proof.** We fix $T > 0$. Let us consider $\widetilde{f}(\boldsymbol{X}) = \frac{f(\boldsymbol{X})}{\exp(-\lambda|\boldsymbol{X}|^2/2)}$ for a positive $\lambda$ to be specified. It is clear that $\widetilde{f}$ is solution of the following variational formulation: $\forall g \in H^1_{\exp(-\Pi)}$,

$$\frac{d}{dt}\int_B \widetilde{f} g \exp(-\widetilde{\Pi}) + \frac{1}{2}\int_B \nabla \widetilde{f} \nabla g \exp(-\widetilde{\Pi}) = \int_B (\boldsymbol{G} + \lambda \mathrm{Id})X.\nabla g\, \widetilde{f} \exp(-\widetilde{\Pi}), \tag{B.132}$$

where $\widetilde{\Pi}(\boldsymbol{X}) = \Pi(\boldsymbol{X}) + \frac{\lambda}{2}|\boldsymbol{X}|^2$. Notice that $H^1_{\exp(-\widetilde{\Pi})} = H^1_{\exp(-\Pi)}$. Let us now introduce a regular function $\alpha$ of time, with positive value, to be précised. Using (B.132), we have:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathcal{B}} |\widetilde{f}-\alpha(t)|^2\, 1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\quad= \int_{\mathcal{B}} \frac{\partial \widetilde{f}}{\partial t}(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi}) - \alpha'(t)\int_{\mathcal{B}}(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\quad= -\frac{1}{2}\int_{\mathcal{B}} \nabla\widetilde{f}\nabla(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\qquad+\int_{\mathcal{B}}(\boldsymbol{G}+\lambda\mathrm{Id})\boldsymbol{X}.\nabla(\widetilde{f}-\alpha(t))\,\widetilde{f}1_{\widetilde{f}\leqq\alpha(t)}, \exp(-\widetilde{\Pi})\\
&\qquad-\alpha'(t)\int_{\mathcal{B}}(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\quad= -\frac{1}{2}\int_{\mathcal{B}} |\nabla(\widetilde{f}-\alpha(t))|^2 1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\qquad+\int_{\mathcal{B}}(\boldsymbol{G}+\lambda\mathrm{Id})\boldsymbol{X}.\nabla(\widetilde{f}-\alpha(t))\,(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\qquad+\alpha(t)\int_{\mathcal{B}}(\boldsymbol{G}+\lambda\mathrm{Id})\boldsymbol{X}.\nabla(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\qquad-\alpha'(t)\int_{\mathcal{B}}(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\quad\leqq -\frac{1}{4}\int_{\mathcal{B}} |\nabla(\widetilde{f}-\alpha(t))|^2 1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\qquad+C(\boldsymbol{G},\lambda)\int_{\mathcal{B}} |\widetilde{f}-\alpha(t)|^2\, 1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\qquad+\alpha(t)\int_{\partial B}(\boldsymbol{G}+\lambda\mathrm{Id})\boldsymbol{X}.\boldsymbol{n}(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\qquad-d\,\alpha(t)\lambda\int_{\mathcal{B}}(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\qquad+\alpha(t)\int_{\mathcal{B}}(\boldsymbol{G}+\lambda\mathrm{Id})\boldsymbol{X}.\nabla\widetilde{\Pi}(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi})\\
&\qquad-\alpha'(t)\int_{\mathcal{B}}(\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)} \exp(-\widetilde{\Pi}).
\end{aligned}
$$

These computations can be justified using the fact that for any real $\beta$, if $g \in H^1_{\exp(-\Pi)}$, then $(g-\beta)\,1_{g\leqq\beta} \in H^1_{\exp(-\Pi)}$, and using classical results for variational formulations of PDEs (for example Lemma 1.1 p. 250 and Lemma 1.2 p. 260 in [31]).

We now choose $\lambda = \sup_{t\in[0,T]} |\boldsymbol{G}(t)^s|$ (see (94)), so that, $\forall t \in [0,T]$, $\forall \boldsymbol{X} \in \mathcal{B}$,

$$(\boldsymbol{G}(t)+\lambda\mathrm{Id})\boldsymbol{X}.\boldsymbol{X} \geqq 0.$$

It is then easy to check that, $\forall \boldsymbol{X} \in \partial\mathcal{B}$, $(\boldsymbol{G}(t)+\lambda\mathrm{Id})\boldsymbol{X}.\boldsymbol{n} \geqq 0$, and $\forall \boldsymbol{X} \in \mathcal{B}$,

$$(\boldsymbol{G}+\lambda\mathrm{Id})\boldsymbol{X}.\nabla\widetilde{\Pi}(\boldsymbol{X}) = (\boldsymbol{G}+\lambda\mathrm{Id})\boldsymbol{X}.\nabla\Pi(\boldsymbol{X}) + \lambda(\boldsymbol{G}+\lambda\mathrm{Id})\boldsymbol{X}.\boldsymbol{X} \geqq 0.$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{B}} |\widetilde{f}-\alpha(t)|^2\, 1_{\widetilde{f}\leqq\alpha(t)}\ \exp(-\widetilde{\Pi})$$
$$\leqq C(\boldsymbol{G},\lambda)\int_{\mathcal{B}} |\widetilde{f}-\alpha(t)|^2\, 1_{\widetilde{f}\leqq\alpha(t)}\ \exp(-\widetilde{\Pi})$$
$$-\big(d\,\alpha(t)\lambda+\alpha'(t)\big)\int_{\mathcal{B}} (\widetilde{f}-\alpha(t))1_{\widetilde{f}\leqq\alpha(t)}\ \exp(-\widetilde{\Pi}).$$

By choosing $\alpha(t)=\min_{\mathcal{B}}(\widetilde{f}(0))\exp(-d\lambda\, t)$ and using the Gronwall Lemma, we can then obtain that $\forall t\in[0,T]$, $\forall \boldsymbol{X}\in\mathcal{B}$,

$$\widetilde{f}(t,\boldsymbol{X})\geqq \min_{\mathcal{B}}(\widetilde{f}(0))\exp(-d\lambda\, t),$$

which yields the lower bound in (B.131). The upper bound is obtained by a similar method. □

**Remark 13** (A rigorous formulation of the no-flux boundary condition)**.** The solution of (B.130) is much more regular than $H^1_{\exp(-\Pi)}$ at any time $t>0$. Using a classical procedure to recover Neumann type boundary conditions, it is then possible to give the following sense to the no-flux boundary condition on the solution $f$ of (B.130): for any function $g\in H^1_{\exp(-\Pi)}$,

$$\int_{\partial\mathcal{B}}\left(\frac{1}{2}\nabla f-\boldsymbol{G}(t)\boldsymbol{X}f\right).\boldsymbol{n}\, g\exp(-\Pi)=0. \tag{B.133}$$

We assume that $f_\infty=\frac{\psi_\infty}{\exp(-\Pi)}$ is a stationary solution in the sense of (B.130) and also satisfies the no-flux boundary condition (B.133), with some $\boldsymbol{G}_\infty$.

It is then possible to rewrite the boundary terms (A.125) in terms of $f$ and $f_\infty$:

$$B_2=\int_{\mathcal{D}}\int_{\partial\mathcal{B}}\left(\frac{1}{2}\nabla f-\boldsymbol{G}(t)\boldsymbol{X}f\right).\boldsymbol{n}\ln\left(\frac{f}{f_\infty}\right)\exp(-\Pi)$$
$$+\frac{1}{2}\int_{\mathcal{D}}\int_{\partial\mathcal{B}}\nabla f.\boldsymbol{n}\exp(-\Pi)-\frac{1}{2}\int_{\mathcal{D}}\int_{\partial\mathcal{B}}\frac{f}{f_\infty}\nabla f_\infty.\boldsymbol{n}\exp(-\Pi)$$
$$-\int_{\mathcal{D}}\nabla_{\boldsymbol{x}}\overline{\boldsymbol{u}}:\int_{\partial\mathcal{B}}\boldsymbol{n}\otimes\boldsymbol{X}f_\infty\exp(-\Pi). \tag{B.134}$$

By considering (B.133) and the fact that constant functions are in $H^1_{\exp(-\Pi)}$, it is now clear that if (A.126) and (A.127) are satisfied, then $B_2=0$, since (A.126)–(A.127) imply that $\ln\left(\frac{f}{f_\infty}\right)\in H^1_{\exp(-\Pi)}$ and $\frac{f}{f_\infty}\in H^1_{\exp(-\Pi)}$. We recall that since $b\geqq 2$, $f\exp(-\Pi)$ and $f_\infty\exp(-\Pi)$ are zero on $\partial\mathcal{B}$.

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